(No) Perfect State Transfer on Trees

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Joint work with Chris Godsil and Henry Liu.
Motivation from quantum computing.

Definition of perfect state transfer (PST), example, some known results.

Our results about PST in trees.

Open problems.
Consider a network of interacting qubits.

This can be seen as a simple graph $G = (V, E)$ where a 2-dim complex vector space is assigned to each vertex.

A state of this system is an assignment of an 1-dim vector space to each qubit.

This corresponds to a 1-dim vector space of the $2^{|V|}$-dim associated to the system.
The subspace associated to this state is spanned by

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
Consider the Pauli matrices
\[ \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Given vertices \( a, b \in V(G) \), we denote
\[ \sigma^x_{ab} = I_2 \otimes \cdots \otimes I_2 \otimes \sigma^x_{a \text{th position}} \otimes I_2 \otimes \cdots \otimes \sigma^x_{b \text{th position}} \otimes \cdots \otimes I_2 \]
and similarly for \( \sigma^y \) and \( \sigma^z \).

Two possible choices for the Hamiltonian of the system
\[ H^{xy} = \sum_{ab \in E(G)} \sigma^x_{ab} + \sigma^y_{ab} \quad \text{or} \]
\[ H^{xyz} = \sum_{ab \in E(G)} \sigma^x_{ab} + \sigma^y_{ab} + \sigma^z_{ab} - I_2|V| \]
Continuous-time quantum walk ($H^{xy}$ model)

The shade represents the probability that the qubit is measured at the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
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Time $t = 6$
Continuous-time quantum walk ($H^{xy}$ model)

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Continuous-time quantum walk ($H^{xy}$ model)

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Time $t = 6$
The shade represents the probability that the qubit is measured at the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
The shade represents the probability that the qubit is measured at the state \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).
Continuous-time quantum walk ($H^{xy}$ model)

The shade represents the probability that the qubit is measured at the state \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

Time \( t = 6 \)
The shade represents the probability that the qubit is measured at the state \((1 0)\).
Continuous-time quantum walk ($H^{xy}$ model)

The shade represents the probability that the qubit is measured at the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Time $t = \hbar$
Continuous-time quantum walk ($H^{xy}$ model)

The shade represents the probability that the qubit is measured at the state $\left( \begin{array}{c} 1 \\ 0 \end{array} \right)$.

Time $t =$
Continuous-time quantum walk ($H^{xy}$ model)

The shade represents the probability that the qubit is measured at the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Time $t =$
The shade represents the probability that the qubit is measured at the state \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).
**Perfect state transfer**

We are interested in a perfect transfer of state, i.e., a time $t$ such that

![Diagram](image)

evolves to

![Diagram](image)
Perfect state transfer

We are interested in a perfect transfer of state, i.e., a time $t$ such that

This is related to the problem of transferring information in certain models of quantum computing.
Given a graph $G$, we define its adjacency and Laplacian matrices as follows:

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L(G) = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$
Suppose the system is initialized with the state of one qubit orthogonal to the states of the other qubits.

In this case, the action of $H^{xy}$ is equivalent to the action of the adjacency matrix of the graph.

The action $H^{xyz}$ is equivalent to the action of the Laplacian matrix of the graph.
Consider the Hamiltonian $H^{xy}$. Its action is equivalent to the action of $A(G)$.

By Schrödinger’s Equation, at time $t \in \mathbb{R}^+$, this means that the evolution will be determined by

$$\exp(itA) = \sum_{k \geq 0} \frac{(it)^k}{k!} A^k.$$ 

Perfect state transfer from $a$ to $b$ at a time $t$ is equivalent to having

$$|\exp(itA)_{a,b}| = 1.$$
Example using the adjacency matrix

Suppose $G = P_2 = \circ \quad \circ$. In this case,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{odd})} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{even})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So it is easy to see that

$$\exp(itA) = \sum_{k \geq 0} \frac{(itA)^k}{k!} = \cos(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

\[ t = 0 \]
Example using the adjacency matrix

Suppose $G = P_2 = \begin{array}{c} a \end{array} \begin{array}{c} b \end{array}$.

In this case,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{odd})} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{even})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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$$t = \frac{\pi}{10}$$
Suppose $G = P_2 = \begin{array}{cc} a & \quad b \\ \end{array}$. In this case,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{odd})} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{even})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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$t = \frac{2\pi}{10}$
Example using the adjacency matrix

Suppose \( G = P_2 = \circ \begin{array}{c} a \\ b \end{array} \circ \). In this case,

\[
A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{odd})} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{even})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

So it is easy to see that

\[
\exp(itA) = \sum_{k \geq 0} \frac{(itA)^k}{k!} = \cos(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
t = \frac{3\pi}{10}
\]
Example using the adjacency matrix

Suppose $G = P_2 = \circ \rightarrow \circ$. In this case,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{odd})} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{even})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So it is easy to see that

$$\exp(itA) = \sum_{k \geq 0} \frac{(itA)^k}{k!} = \cos(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$t = \frac{4\pi}{10}$$
Example using the adjacency matrix

Suppose $G = P_2 = a \quad b$. In this case,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{odd})} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{even})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So it is easy to see that

$$\exp(itA) = \sum_{k \geq 0} \frac{(itA)^k}{k!} = \cos(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$t = \frac{\pi}{2}$$
Example using the adjacency matrix

Suppose $G = P_2 = \begin{array}{c} a \\ \hline \hline b \end{array}$. In this case,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{odd})} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{even})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So it is easy to see that

$$\exp(itA) = \sum_{k \geq 0} \frac{(itA)^k}{k!} = \cos(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$t = \frac{6\pi}{10}$$
Example using the adjacency matrix

Suppose $G = P_2 = \begin{array}{ccc}
  a & \quad & b \\
  \bigcirc & \quad & \bigcirc 
\end{array}$. In this case,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A^{(\text{odd})} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A^{(\text{even})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So it is easy to see that

$$\exp(itA) = \sum_{k \geq 0} \frac{(itA)^k}{k!} = \cos(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$t = \frac{7\pi}{10}$$
**Example using the adjacency matrix**

Suppose $G = P_2 = a \quad b$. In this case,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{odd})} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{even})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So it is easy to see that

$$\exp(itA) = \sum_{k \geq 0} \frac{(itA)^k}{k!} = \cos(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$t = \frac{8\pi}{10}$$
Example using the adjacency matrix

Suppose \( G = P_2 = \circ \circ \). In this case,

\[
A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{odd})} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{even})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

So it is easy to see that

\[
\exp(itA) = \sum_{k \geq 0} \frac{(itA)^k}{k!} = \cos(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
t = \frac{9\pi}{10}
\]
Example using the adjacency matrix

Suppose $G = P_2 = \begin{array}{c} a \\ \hline \end{array} \begin{array}{c} b \\ \hline \end{array}$. In this case,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{odd})} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^{(\text{even})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So it is easy to see that

$$\exp(itA) = \sum_{k \geq 0} \frac{(itA)^k}{k!} = \cos(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
Laplacian case of the example

- Considering the $H^{xyz}$ Hamiltonian, its action is equivalent to the action of the Laplacian matrix.

- In the case where $G = P_2$, we have that
  \[ L = I - A. \]

- Thus
  \[ \exp(itL) = \exp(itI)\exp(-itA) = \exp(it)\exp(-itA). \]

- So perfect state transfer also happens with respect to the Laplacian in this case.

- In general, whenever the graph is regular, the adjacency and Laplacian matrices are essentially the same.

- So the models are only going to give different results in non-regular graphs.
Some History and Known Results

- Properties of $\exp(itH^{xyz})$ were first studied by Farhi and Gutmann in 1998, but not in the context of perfect state transfer.

- Perfect state transfer in linear chains (paths) was first considered by Bose in 2003.

- Christandl and others in 2004 and 2005 showed the following results regarding the $H^{xy}$ Hamiltonian:
  - Perfect state transfer happens in the paths with 2 and 3 vertices.
  - It does not happen in longer paths.
  - If a graph $G$ admits perfect state transfer, then any iterated cartesian power of $G$ admits perfect state transfer (cube, hypercube,...).
Some history and known results

- Perfect state transfer in Cayley graphs for $\mathbb{Z}_n$ was characterized by Bašić and others in 2011 and 2013.

- Perfect state transfer in Cayley graphs for $\mathbb{Z}_2^d$ was characterized by Godsil and others in 2008 and 2011.

- The effect of certain graph operations (joins, products) was studied by Tamon and others in 2009-2012.

- Vinet and Zhedanov in 2011 considered the connection between perfect state transfer in weighted paths and orthogonal polynomials.
Our question

- Which trees admit perfect state transfer? That is, for which trees $T$ there are vertices $a$ and $b$ such that, for some $t \in \mathbb{R}^+$,

\[ |\exp(itA(T))_{a,b}| = 1 \quad \text{or} \quad |\exp(itL(T))_{a,b}| = 1 \quad ? \]

- In the first case, we know it happens between the extreme vertices of the paths

  ![Diagram of two paths with four vertices each]

- In the second case, we know it happens in the path with two vertices.

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Our question

For example, could it be that something like

admits perfect state transfer?
What is the minimum number of edges needed to achieve perfect state transfer at distance $d$?

Best known: in the $k$th iterated cartesian power of $P_3$, we can achieve perfect state transfer at distance $2k$ with $2k3^{k-1}$ edges.
**Our results**

**Theorem**
*Except for the path on two vertices, no tree admits Laplacian perfect state transfer.*

**Theorem**
*Except for the path on two vertices, no tree that contains a perfect matching admits adjacency perfect state transfer.*
Our results

**Theorem**

Except for the path on two vertices, no tree admits Laplacian perfect state transfer.

**Theorem**

Except for the path on two vertices, no tree that contains a perfect matching admits adjacency perfect state transfer.
Let $L$ be the Laplacian matrix of a tree $T$. It is positive semi-definite, and 0 is always an eigenvalue.

Because $L$ is symmetric, it has real (distinct) eigenvalues $0 = \lambda_0 < \ldots < \lambda_d$, and there is a spectral decomposition into orthogonal projectors

$$L = \sum_{r=0}^{d} \lambda_r F_r.$$
No Laplacian PST on trees

- It follows that
  \[ \exp(itL) = \sum_{r=0}^{d} e^{it\lambda_r} F_r. \]

- Thus, for some \( t \in \mathbb{R}^+ \),
  \[ |\exp(itL)_{a,b}| = 1 \]
  if and only if
  - The \( a \)-th column of \( F_r \) is \( \pm \) the \( b \)-th column of \( F_r \).
  - If the \( a \)-th column of \( F_r \) is non-zero, then \( \lambda_r \) is integer.
  - Some parity conditions on these integer eigenvalues.
If $T$ is tree with $n$ vertices, the so-called Matrix Tree Theorem implies that any $(n-1) \times (n-1)$ minor of $L$ is equal to 1.

Thus the rank of $L(T)$ over any field of prime order is $n-1$.

As a consequence, if $L \mathbf{v} = \lambda \mathbf{v}$ with $\lambda \in \mathbb{Z}$ and $\mathbf{v} \in \mathbb{Z}^n$, and if $p$ divides $\lambda$, we have that

$$\mathbf{v} \equiv (\text{all 1s vector}) \pmod{p}.$$ 

Thus two entries of $\mathbf{v}$ cannot be $(-1)$ times each other.
Coupled with the parity conditions and some other observations, this restricts Laplacian perfect state transfer in trees with more than three vertices to situations as follows:
But then we showed that

**Theorem**

*In any graph $G$, Laplacian perfect state transfer never happens between vertices $a$ and $b$ of the following form*
Our methods could also be used to show that:

- No Laplacian PST in graphs with an odd number of vertices and odd number of spanning trees.
  - 853 graphs on 7 vertices, 339 have odd number of spanning trees.

- If Laplacian PST happens in a bipartite graph, the largest Laplacian eigenvalue must be an integer.
  - 182 bipartite graphs on 8 vertices, only 10 have an integral largest eigenvalue.
Laplacian perfect state transfer is related to the number of spanning trees of the graph.

The number of spanning trees is an evaluation of the Tutte polynomial - related to the partition function of some spin models.

The number of spanning trees is also the size of the critical group of the graph.
Possibly show that no trees (except for $P_2$ and $P_3$) admit perfect state transfer in the adjacency matrix model.

Can we find a family of simple graphs in which perfect state transfer happens between vertices at distance $d$ and the number of edges is linear in $d$? Polynomial in $d$?