DCC888 – Twelf [1]

Name: ______________________________ ID: ____________

1. We can define the even numbers in Twelf in the following way:

```
nat : type.
z   : nat.
s   : nat -> nat.
```

```
even   : nat -> type.
even-z : even z.
even-s : even (s (s N))
    <- even N.
```

The rest of this question is about the definitions above.

   (a) Define the `odd` numbers with a judgment analogous to that one used to define the `even` numbers above.

   (b) Prove the following relation between `odd` and `even` numbers: "the successor of an even number is an odd number". You can state this theorem, in Twelf, as follows:

```
succ-even : even N -> odd (s N) -> type.
%mode succ-even +D1 -D2.
%worlds () (succ-even _ _).
%total D (succ-even D _).
```

   (c) What is the meaning of `D` in the clause `%total D (succ-even D _)`?

[1] These exercises have been taken from the Twelf web-page, available at `http://twelf.org/`

(d) Now prove the inverse relation: "the successor of an odd number is an even number". You can state this theorem as follows:

```
succ-odd : odd N -> even (s N) -> type.
%mode succ-odd +D1 -D2.
%worlds () (succ-odd _ _).
%total D (succ-odd D _).
```

2. Addition of natural numbers, in Twelf, can be defined by the following judgments:

```
plus   : nat -> nat -> nat -> type.
plus-z : plus z N2 N2.
plus-s : plus (s N1) N2 (s N3)
    <- plus N1 N2 N3.
```

As we all know, addition is commutative. We would like to state this fact as a theorem, and prove it in Twelf. However, it is easier if we state a few lemmas before, that can help us in our proof.

(a) The first lemma says that for any natural number $n, n + 0 = n$. We state this lemma as follows:

```
plus-zero-id :  {N1 : nat} plus N1 z N1 -> type.
%mode plus-zero-id +N -D.
%worlds () (plus-zero-id _ _).
%total N (plus-zero-id N _).
```

Prove this lemma in Twelf.

(b) The second lemma states that if $n_1 + n_2 = n_3$, then we have that $n_1 + \text{succ}(n_2) = \text{succ}(n_3)$. Give a proof, in Twelf, that this lemma is true. We can state it as follows:

```
plus-flip : plus N1 N2 N3 -> plus N1 (s N2) (s N3) -> type.
%mode plus-flip +D1 -D2.
%worlds () (plus-flip _ _).
%total D (plus-flip D _).
```

(c) With these two lemmas, we can finish easily the proof of the commutativity of addition. It goes as follows:

```
plus-commutes : plus N1 N2 N3 -> plus N2 N1 N3 -> type.
pcz : plus-commutes _ D <- plus-zero-id N1 D.
pcs : plus-commutes (plus-s Dplus: plus (s N1') N2 (s N3')) D
        <- plus-commutes Dplus DIH
        <- plus-flip DIH D.
%mode plus-commutes +D1 -D2.
%worlds () (plus-commutes _ _).
%total N (plus-commutes N _).
```

Explain the rule `pcz`. Why is it not necessary to specify the Rule `plus-z` to indicate that we are talking about a sum involving zero?

(d) Now, explain the meaning of the syntax `Dplus: plus (s N1') N2 (s N3')`, in the proof for the case `pcs` above.

3. The sum of an even and an odd number is always odd. We can state this fact as a Twelf theorem as follows:

```
sum-even-odd : even N1 -> odd N2 -> plus N1 N2 N3 -> odd N3 -> type.
%mode sum-even-odd +D1 +D2 +D3 -D4.
%worlds () (sum-even-odd _ _ _ _).
%total D (sum-even-odd D _ _ _).
```

The rest of this question is about this fact.

(a) Why is this theorem true if the first argument of the sum, e.g., `N1`, is zero? In other words, show – on the paper, not in Twelf – that if we have $n_1 + n_2 = n_3$, $n_1 = 0$, and n_2 odd, then n_3 is odd.

(b) Now, write this proof in Twelf. It is an easy translation of the reasoning that you presented above.

(c) If, on the other hand, we have that $n_1 + n_2 = n_3$, $n_1 \neq 0$, but n_1 even, and n_2 odd, then we must use some inductive argument to conclude the proof. Why is the lemma true when $n_1 \neq 0$? Explain it using inference rules. Try to show which facts are true by the premises of the theorem, and which facts you must conclude by induction.

(d) Write your proof of the last fact in Twelf.