On sesqui-regular graphs with fixed smallest eigenvalue

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Online seminar in Algebraic graph theory

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(Joint work with Jack Koolen, Brhane Gebremichel and Jae Young Yang)

Notations

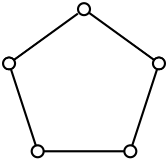
Let G be a finite, undirected and simple graph.

- vertex set: $V(G)$, edge set: $E(G)$
- adjacency matrix: $A(G)$
- smallest eigenvalue $\lambda_{\min}(G)$: the smallest eigenvalue of $A(G)$
- $H \subseteq G$: H is an induced subgraph of G.
- $H \not\subseteq G$: H is not an induced subgraph of G.
- If every vertex of G has valency k, then G is k-regular.

Definition

A graph G is strongly-regular with parameters (v, k, a, c), if

1. it has v vertices,
2. it is k-regular,
3. every pair of **adjacent vertices** has exactly a common neighbors,
4. every pair of **distinct non-adjacent vertices** has exactly c common neighbors.



strongly-regular graph : $(5, 2, 0, 1)$

SRGs with fixed smallest eiganvalue

Let $\lambda \geq 2$ be an integer. If G is a connected strongly-regular graph with parameters (v, k, a, c) and $\lambda_{\min}(G) = -\lambda$, then

Result 1

1. G is a complete multipartite graph, or
2. $c \leq \lambda^3(2\lambda - 3)$.

Result 2

1. G is a complete multipartite graph, a Steiner graph, a Latin square graph, or
2. G is in a finite set of strongly regular graphs.

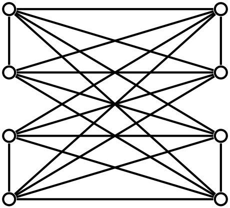
A. Neumaier, Strongly regular graphs with smallest eigenvalue m, *Arch. Math.(Basel)*, 33(4):392–-400, 1978–80.

A. E. Brouwer, H. van Maldeghem, *Strongly regular graphs*, Cambridge University Press, Cambridge, 2022.

Definition

A graph G is co-edge-regular with parameters (v, k, c), if

1. it has v vertices,
2. it is k-regular,
3. every pair of **distinct non-adjacent vertices** has exactly c common neighbors.



co-edge-regular: $(8, 5, 4)$

Co-edge-regular graphs with fixed smallest eiganvalue

Let $\lambda \geq 2$ be a real number. There exists a constant $C_1(\lambda)$ such that, if G is a connected co-edge-regular graph with parameters (v, k, c) and $\lambda_{\min}(G) \geq -\lambda$, then

Result

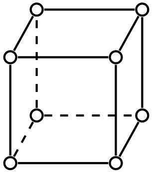
1. $v - k - 1 \leq \frac{(\lambda-1)^2}{4} + 1$, or
2. $c \leq C_1(\lambda)$.

J. Y. Yang, J. H. Koolen, On the order of regular graphs with fixed second largest eigenvalue, *Linear Algebra Appl.*, 610:29–39, 2021.

Definition

A graph G is sesqui-regular with parameters (v, k, c), if

1. it has v vertices,
2. it is k-regular,
3. every pair of **vertices of distance** 2 has exactly c common neighbors.



sesqui-regular graph: $(8, 3, 2)$

Sesqui-regular graphs with fixed smallest eiganvalue

Let $\lambda \geq 2$ be a real number. There exists a constant $C_2(\lambda)$ such that, if G is a connected sesqui-regular graph with parameters (v, k, c) and $\lambda_{\min}(G) \geq -\lambda$, then

Result 1

1. $v - k - 1 \leq \frac{(\lambda-1)^2}{4} + 1$, or
2. $c \leq C_2(\lambda)$.

Proof: It can be proven by the same method as for co-edge-regular graphs. See the reference below.

J. Y. Yang, J. H. Koolen, On the order of regular graphs with fixed second largest eigenvalue, *Linear Algebra Appl.*, 610:29–39, 2021.

Sesqui-regular graphs with fixed smallest eiganvalue

Let $\lambda \geq 2$ be an integer. There exists a constant $\kappa_1(\lambda)$ such that, if G is a connected sesqui-regular graph with parameters (v, k, c), where $k \geq \kappa_1(\lambda)$, and $\lambda_{\min}(G) \geq -\lambda$, then

Result 2 (Main result)

1. $v - k - 1 \leq \frac{(\lambda-1)^2}{4} + 1$, or
2. $c \leq \lambda^2(\lambda - 1)$.

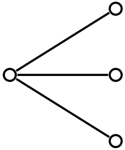
J. H. Koolen, B. Gebremichel, J. Y. Yang, Q. Yang, Sesqui-regular graphs with fixed smallest eigenvalue, submitted to *Linear Algebra Appl.*, 2021.

Two important graphs

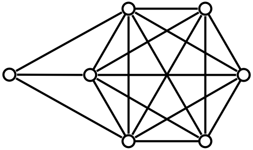
Let $\lambda \geq 2$ be an integer. Let G be a graph with $\lambda_{\min}(G) \geq -\lambda$. If $\lambda_{\min}(H) < -\lambda$, then $H \not\subseteq G$.

$K_{1,t}$ and $\widetilde{K}_{2m}$

- $\lambda_{\min}(K_{1,t}) = -\infty$ if $t \to \infty$.
- Let $\widetilde{K}_{2m}$ be the graph on $2m+1$ vertices consisting of a clique K_{2m} together with a vertex that is adjacent to precisely m vertices of this clique. $\lambda_{\min}(\widetilde{K}_{2m}) = -\infty$ if $m \to \infty$.



$K_{1,3}$



$\widetilde{K}_6$

Let $\lambda \geq 2$ be an integer. Let G be a k-regular graph with $\lambda_{\min}(G) \geq -\lambda$.

Associated Hoffman graph

- $\exists\ t(\lambda)$ s.t. $K_{1,t} \not\subseteq G$ for any $t \geq t(\lambda)$.
  $\Longrightarrow$ If k is very large, then every vertex is in a "large" clique.
  (Ramsey theory)

Let $\lambda \geq 2$ be an integer. Let G be a k-regular graph with $\lambda_{\min}(G) \geq -\lambda$.

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  $\Longrightarrow$ If k is very large, then every vertex is in a "large" clique. (Ramsey theory)
- $\exists\ m(\lambda)$ s.t. $\widetilde{K}_{2m} \nsubseteq G$ for any $m \geq m(\lambda)$.
  $\Longrightarrow$ There is an equivalence relation $\equiv_n^m$ on the set

  $$\mathcal{C}(n) = \{C \mid C \text{ is a maximal clique of } G \text{ of order } \geq n\}$$

  for any $n \geq (m+1)^2$.

  $$C_1 \equiv_n^m C_2,\ \forall\ C_1, C_2 \in \mathcal{C}(n)$$
  $$\Updownarrow$$
  every vertex of C_1 has at most $m-1$ non-neighbors in C_2

- Let $C \in \mathcal{C}(n)$. The quasi-clique $Q([C]_n^m)$, with respect to $[C]_n^m$ and the pair (m, n), is the subgraph induced on

$$\{x \in V(G) \mid x \text{ has at most } m-1 \text{ non-neighbors in } C\}.$$

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- Assume $\widetilde{K}_{2m} \not\subset G$ and $n \geq (m+1)^2$. Let $[C_1]_n^m$, $[C_2]_n^m$, ..., $[C_t]_n^m$ be all the equivalence classes under $\equiv_n^m$. Attach "fat" vertices $f_1, \ldots, f_t$ to G s.t.
  1. $f_1, \ldots, f_t$ are mutually non-adjacent,
  2. f_i is adjacent to every vertex in quasi-clique $Q([C_i]_n^m)$, for $i = 1, ..., t$.

  The resulting graph is the associated Hoffman graph of G with respect to (m, n), and denoted by $\mathfrak{g}(G, m, n)$.

Definition

A Hoffman graph $\mathfrak{h}$ is a pair (H, μ) of a graph $H = (V, E)$ and a labeling map $\mu : V \to \{f, s\}$ s.t.

1. every vertex with label f is adjacent to at least one vertex with label s,
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- A vertex with label s called a slim vertex.
  A vertex with label f called a fat vertex.
- If every slim vertex has a fat neighbor, we call $\mathfrak{h}$ fat.

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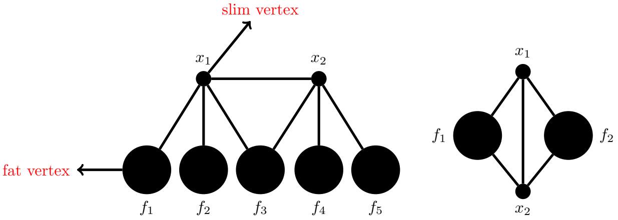

Assume $\mathfrak{h} = (H, \mu)$ is a Hoffman graph. Let

$$V_s(\mathfrak{h}) := \{x \in V(H) \mid \mu(H) = s\}.$$

Special matrix

The special matrix $Sp(\mathfrak{h})$ of $\mathfrak{h}$ is the matrix whose rows and columns are indexed by $V_s(\mathfrak{h})$ s.t.

$$Sp(\mathfrak{h})_{x,y} = \begin{cases} -|\{z \mid z \text{ is a fat neighbor of } x\}| & x = y, \\ 1 - |\{z \mid z \text{ is a fat neighbor of both } x \text{ and } y\}| & x \sim y, \\ -|\{z \mid z \text{ is a fat neighbor of both } x \text{ and } y\}| & \text{otherwise.} \end{cases}$$

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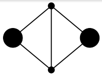
For the Hoffman graph $\mathfrak{h}_1 =$  ,

$$Sp(\mathfrak{h}_1) = \begin{pmatrix} -2 & 1-2 \\ 1-2 & -2 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}.$$

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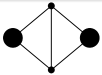
For the Hoffman graph $\mathfrak{h}_1 =$  ,

$$Sp(\mathfrak{h}_1) = \begin{pmatrix} -2 & 1-2 \\ 1-2 & -2 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}.$$

- The eigenvalues of $\mathfrak{h}$ are the eigenvalues of $Sp(\mathfrak{h})$.
- $\lambda_{\min}(\mathfrak{h})$: the smallest eigenvalue of $\mathfrak{h}$.

Property of a special matrix

Let $\mathfrak{h} = (H, \mu)$ be a Hoffman graph.

- If $\mathfrak{h}$ has no fat vertices, then $\mathfrak{h}$ is a graph, and its special matrix and adjacency matrix are the same.
- If $\mathfrak{h}' \subseteq \mathfrak{h}$ is an induced Hoffman subgraph, then

$$\lambda_{\min}(\mathfrak{h}') \geq \lambda_{\min}(\mathfrak{h}).$$

- (Hoffman-Ostrowski theorem) Let $G(\mathfrak{h}, n)$ be the ordinary graph obtained from $\mathfrak{h}$ by replacing each fat vertex f by an n-clique $K_n(f)$, and joining all the neighbors of f with all the vertices of $K_n(f)$. Then

$$\lambda_{\min}(G(\mathfrak{h}, n)) \geq \lambda_{\min}(\mathfrak{h}),$$

and

$$\lim_{n\to\infty} \lambda_{\min}(G(\mathfrak{h}, n)) = \lambda_{\min}(\mathfrak{h}).$$

Property of associated Hoffman graph

- Let $\lambda \geq 1$ be a real number.
- Let $\mathcal{H}$ be a finite family of Hoffman graphs with smallest eigenvalue $< -\lambda$.
- Let $m = \min\{m' \mid \lambda_{\min}(\widetilde{K}_{2m'}) < -\lambda\}$.

There exists an integer $N(m, \mathcal{H}) \geq (m+1)^2$, s.t. for any graph G with $\lambda_{\min}(G) \geq -\lambda$ and any integer $n \geq N(m, \mathcal{H})$, the associated Hoffman graph $\mathfrak{g}(G, m, n)$ does not contain any Hoffman graph in $\mathcal{H}$ as an induced Hoffman subgraph.

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Now, we are going to show the sketch of the proof for main result.

Main result

Let $\lambda \geq 2$ be an integer. There exists a constant $\kappa_1(\lambda)$ such that, if G is a connected sesqui-regular graph with parameters (v, k, c), where $k \geq \kappa_1(\lambda)$, and $\lambda_{\min}(G) \geq -\lambda$, then

1. $v - k - 1 \leq \frac{(\lambda-1)^2}{4} + 1$, or
2. $c \leq \lambda^2(\lambda - 1)$.

Main result

Let $\lambda \geq 2$ be an integer. There exists a constant $\kappa_1(\lambda)$ such that, if G is a connected sesqui-regular graph with parameters (v, k, c), where $k \geq \kappa_1(\lambda)$, and $\lambda_{\min}(G) \geq -\lambda$, then

1. $v - k - 1 \leq \frac{(\lambda-1)^2}{4} + 1$, or
2. $c \leq \lambda^2(\lambda - 1)$.

Known result

Let $\lambda \geq 2$ be a real number. There exists a constant $C_2(\lambda)$ such that, if G is a connected sesqui-regular graph with parameters (v, k, c) and $\lambda_{\min}(G) \geq -\lambda$, then

1. $v - k - 1 \leq \frac{(\lambda-1)^2}{4} + 1$, or
2. $c \leq C_2(\lambda)$.

So we may assume $c \leq C_2(\lambda)$, and prove $c \leq \lambda^2(\lambda - 1)$ when k is sufficiently large.

Proof

Let $m = \min\{m' \mid \lambda_{\min}(\widetilde{K}_{2m'}) < -\lambda\}$

Step 1 As $c \leq C_2(\lambda)$, there exists an integer N_1 s.t. for any integer $n \geq N_1$, every quasi-clique in the associated Hoffman graph $\mathfrak{g}(G, m, n)$ is a clique.

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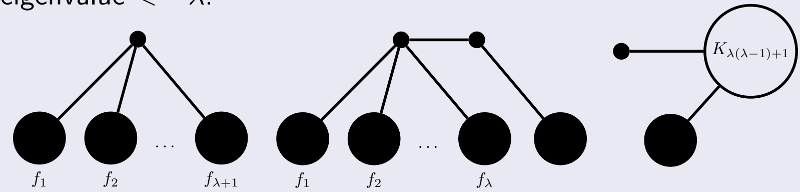
Step 1 As $c \leq C_2(\lambda)$, there exists an integer N_1 s.t. for any integer $n \geq N_1$, every quasi-clique in the associated Hoffman graph $\mathfrak{g}(G, m, n)$ is a clique.

Step 2 There exists an integer N_2 s.t. for any integer $n \geq N_2$, the associated Hoffman graph $\mathfrak{g}(G, m, n)$ does not contain the following as an induced Hoffman subgraph, as they have smallest eigenvalue $< -\lambda$.



Proof

Let $t = \min\{t' \mid \lambda_{\min}(K_{1,t'}) < -\lambda\}$ and $N = \max\{N_1, N_2\}$.

Step 3 Let $k \geq R(N, t)$. Then every vertex of G is in a clique of order at least N, that is, in the associated Hoffman graph $\mathfrak{g}(G, m, N)$, every slim vertex is in a quasi-clique.

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Let $t = \min\{t' \mid \lambda_{\min}(K_{1,t'}) < -\lambda\}$ and $N = \max\{N_1, N_2\}$.

Step 3 Let $k \geq R(N, t)$. Then every vertex of G is in a clique of order at least N, that is, in the associated Hoffman graph $\mathfrak{g}(G, m, N)$, every slim vertex is in a quasi-clique.

Look at the associated Hoffman graph $\mathfrak{g}(G, m, N)$.

Step 4 Let $x \in V(G) = V_s(\mathfrak{g}(G, m, N))$.

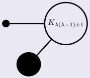
Case 1 Every (slim) neighbor of x is in a same quasi-clique as x.

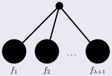
Case 2 There exists a (slim) neighbor of x which is not in a same quasi-clique as x.

Proof

Case 1 Every (slim) neighbor of x is in a same quasi-clique as x.

Assume $\text{distance}(x, y) = 2$ in G.

- Step 1 says that x and y are not in the same quasi-clique in $\mathfrak{g}(G, m, N)$.
- 

$\not\subseteq \mathfrak{g}(G, m, N)$ implies that y has at most $\lambda(\lambda - 1)$ slim neighbors in each quasi-clique containing x.
- 

$\not\subseteq \mathfrak{g}(G, m, N)$ implies that there are $\leq \lambda$ quasi-cliques containing x.

Thus, x and y have at most $\lambda * \lambda(\lambda - 1)$ common slim neighbors.

Proof

Case 2 There exists a (slim) neighbor of x which is not in a same quasi-clique as x.

Let

$$S = \{z \mid z \text{ is a slim neighbor of } x \text{ and is not in a same quasi-clique as } x\}.$$

- There exists a vertex $w \in V(G) = V_s(\mathfrak{g}(G, m, N))$ s.t.
  - $\mathrm{distance}(x, w) = 2$ in G,
  - The common slim neighbors of x and w in S are contained in a quasi-clique Q,
  - w is in Q.

Proof

Case 2 There exists a (slim) neighbor of x which is not in a same quasi-clique as x.

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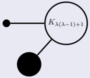
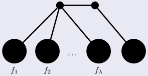
  - w is in Q.
- From Step 1, x is not in Q. 

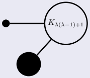
$\not\subseteq \mathfrak{g}(G, m, N)$ implies that x has at most $\lambda(\lambda - 1)$ slim neighbors in Q.
- Inside S, x and w have at most $\lambda(\lambda - 1)$ common slim neighbors.

Proof

Step 4 Let $x \in V(G) = V_s(\mathfrak{g}(G, m, N))$.

Case 2 There exists a (slim) neighbor of x which is not in a same quasi-clique as x.

- 

$\not\subseteq \mathfrak{g}(G, m, N)$ implies that there are $\leq \lambda - 1$ quasi-cliques containing x.
- 

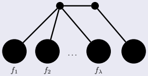
$\not\subseteq \mathfrak{g}(G, m, N)$ implies that w has at most $\lambda(\lambda - 1)$ slim neighbors in each quasi-clique containing x.
- Outside S, x and y have at most $(\lambda - 1) * \lambda(\lambda - 1)$ common slim neighbors.

Thus, x and w have at most $\lambda^2(\lambda - 1)$ common slim neighbors.

Proof

Step 4 Let $x \in V(G) = V_s(\mathfrak{g}(G, m, N))$.

Case 2 There exists a (slim) neighbor of x which is not in a same quasi-clique as x.

- 

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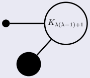
- 

$\not\subseteq \mathfrak{g}(G, m, N)$ implies that w has at most $\lambda(\lambda - 1)$ slim neighbors in each quasi-clique containing x.
- Outside S, x and y have at most $(\lambda - 1) * \lambda(\lambda - 1)$ common slim neighbors.

Thus, x and w have at most $\lambda^2(\lambda - 1)$ common slim neighbors.

Conclusion: $c \leq \lambda^2(\lambda - 1)$.

$\lambda = 3$

Let $\lambda \geq 2$ be an integer.

Main result

There exists a constant $\kappa_1(\lambda)$ such that, if G is a connected sesqui-regular graph with parameters (v, k, c), where $k \geq \kappa_1(\lambda)$, and $\lambda_{\min}(G) \geq -\lambda$, then

1. $v - k - 1 \leq \frac{(\lambda-1)^2}{4} + 1$, or
2. $c \leq \lambda^2(\lambda - 1)$.

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2. $c \leq \lambda^2(\lambda - 1)$.

Let $\lambda = 3$.

There exists a constant $\kappa_1(3)$ such that, if G is a connected sesqui-regular graph with parameters (v, k, c), where $k \geq \kappa_1(3)$, and $\lambda_{\min}(G) \geq -3$, then

1. $v - k - 1 \leq 2$, or
2. $c \leq 18$.

$\lambda = 3$

There exists a positive integer κ_2 such that, if G is a connected sesqui-regular graph with parameters (v, k, c), where $k \geq \kappa_2$, and smallest eigenvalue $\lambda_{\min}(G) \in [-3, -2)$, then

1. $c = k$ and G is the complete multipartite graph $K_{\frac{n}{3}\times 3}$, that is, the complement of the disjoint union of cycles with length 3,
2. $c = k - 1$ and G is the complement of the disjoint union of cycles with length at least 4,
3. $c = 9$ and G is a Steiner graph (G is strongly-regular),
4. $c \leq 8$.

Q. Yang, B. Gebremichel, M. U. Rehman, J. Y. Yang, J. H. Koolen, Sesqui-regular graphs with smallest eigenvalue at least -3, arxiv:2109.03491, 2021.

Questions

Question 1

What is the classification for sesqui-regular graphs with parameters (v, k, c) and smallest eigenvalue ≥ -3, where k is sufficiently large and $c \leq 8$?

Question 2

What are the applications of Hoffman graphs to graphs with fixed smallest eigenvalue and good combinatorial structure?

Thank you for your attention!