Assignment 4 - due April 4th

Send answers to `dcc030ufmg@gmail.com`.

Choose 3 exercises (but I encourage you to do all 5 of them).

Exercise 1 (2.39 in the notes)**.** Let G be a graph with largest eigenvalue λ and largest degree Δ. Prove that $\lambda \geq \sqrt{\Delta}$.

Exercise 2. Suppose $\mathbf{M}$ is a symmetric, non-zero, non-negative and irreducible matrix, and $\mathbf{D}$ is an arbitrary diagonal matrix (possibly with negative entries). Let $\mathbf{N} = \mathbf{M} + \mathbf{D}$. Prove that $\mathbf{N}$ has an eigenvector with all entries positive, whose corresponding eigenvalue is larger than the smallest entry of $\mathbf{D}$.

Exercise 3. Let G be a connected graph and H be any proper subgraph of G (obtained from removing at least one edge or at least one vertex of G). Show that the largest eigenvalue of $\mathbf{A}(G)$ is strictly larger than the larger eigenvalue of $\mathbf{A}(H)$.

Exercise 4 (2.45 in the notes)**.** Prove that there is no connected regular graph (other than the triangle C_3) in which any two vertices share precisely one common neighbour.

(If you are feeling brave, try to show that the only graphs satisfying the property that any two vertices share precisely one common neighbour are formed by glueing triangles at one single vertex, neighbour of all other vertices of the graph. To do that, assume there is no vertex who is a neighbour of all vertices and that the graph is not regular, and arrive at a contradiction. Then use the result you proved above. This part in parenthesis is an extra exercise, and will make no use of spectral stuff. Only combinatorial arguments.)

Exercise 5. Assume G is connected and regular, and that G has exactly four distinct eigenvalues. For any $k \geq 0$, prove that the number of closed walks of length k that start and end at a vertex v is the same for all vertices v of the graph.