Assignment 7 - due May 16th

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Given a graph $G = (V, E)$, its line graph $L(G)$ is a graph whose *vertex set* is E, and two of these vertices are adjacent precisely if the corresponding edges in G meet at a vertex. For example, $L(P_n) = P_{n-1}$, $L(K_3) = K_3$ and $L(S_n) = K_n$, where S_n stands for the star with n leaves. The incidence matrix $\mathbf{N}$ of G is a $|V| \times |E|$ matrix, with 1s and 0s indicating whether the vertex of the row is incident or not to the edge of the column.

Exercise 1.

(a) For any graph G, argue why $\mathbf{A}(L(G)) = \mathbf{N}^T\mathbf{N} - 2\mathbf{I}$.

(b) If G is k-regular, explain why $\mathbf{A}(G) = \mathbf{N}\mathbf{N}^T - k\mathbf{I}$.

(c) Now let G be k-regular, with eigenvalues $\theta_0 > ... > \theta_d$. Describe the spectrum of $\mathbf{A}(L(G))$ in terms of these eigenvalues.

Consider now the Petersen graph, which has eigenvalues $3, 1^{(5)}, -2^{(4)}$. If the Petersen graph contains a Hamilton cycle, then convince yourself of the fact that its line graph contains an induced cycle C_{10} (induced means you will pick 10 vertices and the edges of C_{10} are the only edges between). Recall from the notes the eigenvalues of C_{10}.

Exercise 2. Use interlacing now to show that the Petersen graph does not have a Hamilton cycle.

Exercise 3. Let G be k-regular on n vertices, with eigenvalues $\theta_1 \geq ... \geq \theta_n$. Assume G contains an induced subgraph H with n' vertices and m' edges. Show that

$$\theta_2 \geq \frac{2m'n - (n')^2k}{n'(n - n')} \geq \theta_n.$$

Characterize what happens if equality holds in either side.