# Assignment 1 - due March 11th

Send pdf with answers to `dcc030ufmg@gmail.com`. They must be typed up in latex.

- ABOUT COLLABORATION AND/OR RESEARCH
  (THESE ARE RULES FOR ALL ASSIGNMENTS)
  - If the question has a $*$, you should not collaborate or research the question in sources other than the course notes. You can ask me questions about them though.
  - For the other questions, you can talk to your colleagues — in this case, list the names of all them (even if you only helped them) and indicate if any of them gave a decisive contribution; or you can research elsewhere — in which case indicate the source.
  - Even if you collaborate or research, try to write your own solution after the discussion or after reading the source. Do not write your solutions while looking at someone else's or at a source. Do not send your written up solutions to anyone!
  - As long as you are trying to do some questions on your own, your write-ups look original and you are giving due credit, your final grade will not be affected by having done some collaboration or research.
  - It is not a big problem if you do not do some letters. I prefer you do fewer questions well, rather than all questions badly.

**Exercise 1 ($*$).** Prove that the eigenvalues of unitary matrices are complex numbers of absolute value equal to 1.

**Exercise 2 ($*$).** Assume $\mathbf{M} \succcurlyeq \mathbf{0}$. Show that $\mathbf{v}^*\mathbf{M}\mathbf{v} = \mathbf{0}$ if and only if $\mathbf{M}\mathbf{v} = \mathbf{0}$.

**Exercise 3 ($*$).** Assume square matrices $\mathbf{A}$ and $\mathbf{B}$ are diagonalizable. Show that $\mathbf{A}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{B}$ is diagonalizable, and express its eigenvalues in terms of those of $\mathbf{A}$ and $\mathbf{B}$. (Here $\mathbf{I}$ means the identity matrix of convenient size).

**Exercise 4.** Prove that $\mathbf{M}$ is a normal matrix if and only if $\mathbf{M}^*$ is a polynomial in $\mathbf{M}$.

**Exercise 5.** (a) Prove that two Hermitian matrices $\mathbf{M}$ and $\mathbf{N}$ commute if and only if they can be simultaneously diagonalized by the same set of orthonormal eigenvectors.

(b) Consider the spectral decomposition $\mathbf{M} = \sum_{i=1}^{d} \lambda_r \mathbf{F}_r$, with $\lambda_r \neq \lambda_s$ for $r \neq s$. Describe explicitly a polynomial $p_r(x)$ so that $p_r(\mathbf{M}) = \mathbf{F}_r$.

(c) Prove that $\mathbf{N}$ commutes with each matrix that commutes with $\mathbf{M}$ if and only if $\mathbf{N}$ is a polynomial in $\mathbf{M}$.

**Exercise 6.** Assume $\mathbf{P}_i$s are orthogonal projections.

(a) Show that $\mathbf{P}_1 + \mathbf{P}_2$ is an orthogonal projection if and only if $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$.

(b) Show now that $\mathbf{P}_1 + ... + \mathbf{P}_k$ is an orthogonal projection if and only if $\mathbf{P}_i\mathbf{P}_j = \mathbf{0}$ for $i \neq j$. (I will accept unhappily if you do the difficult direction only for $k = 3$).

(c) (Bonus) Assume now only that $\mathbf{P}_i$s are $n \times n$ positive semidefinite matrices such that $\text{rank}(\mathbf{P}_i) = r_i$ and $\sum_i r_i = n$. If $\mathbf{P}_1 + \cdots + \mathbf{P}_k = \mathbf{I}$, prove that $\mathbf{P}_i\mathbf{P}_j = \delta_{ij}\mathbf{P}_i$, where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise.