Assignment 12 - due November 2nd

Send pdf with answers to `dcc030ufmg@gmail.com`. They must be typed up in latex.

This assignment is about Chapter 13 of the book Graph Spectra and Quantum Walks, and a bit about Chapter 9.

1st part

- Sections 13.8 to 13.12 are short, and discuss some examples of how to compute the average mixing matrix of certain classes of graphs. You can check them quickly to get a good intuition.
- The goal of this assignment is to understand well states given by density matrices and their averages. Read carefully Section 13.14, and 9.1, 9.4 and 9.5.

2nd part

Exercise 1. Compute the average mixing matrix for the complete graphs.

Exercise 2. Theorem 13.14.1 contains a mistake. Fix it, by showing explicitly that the average mixing matrix of X is the Gram matrix (according to the inner product $\langle M, N\rangle = \operatorname{tr} M^T N$) of the matrices $\Psi(D_a)$, for $a \in V(X)$.

Exercise 3. Prove explicitly that, for all square matrices M and N, $\langle \Psi(M), N - \Psi(N)\rangle = 0$ (thus observing the geometric notion of Ψ being an orthogonal projection).

Exercise 4. Let A be a real symmetric matrix. What is the dimension of $\operatorname{comm}(A)$, as a vector space over $\mathbb{R}$, and in terms of the multiplicity of the eigenvalues of A? (Hint: Start showing that a matrix M commutes with A iff the eigenspaces of A are M-invariant).

Exercise 5. Show Lemma 13.14.4, that says a and b are strongly cospectral if and and only if $\Psi(D_a) = \Psi(D_b)$.

Exercise 6. The goal of this exercise is to compute $\operatorname{rank}(\widehat{M_X})$, the rank of the average mixing matrix, for when X is a cycle C_n (without using the explicit computation of 13.10).

(a) Consider the map that takes a diagonal matrix D, applies Ψ, and then extracts its diagonal. That is, $D \mapsto \Psi(D) \circ I$. Show that $\widehat{M}$, the average mixing matrix of X, is the matrix that represents this map according to the basis $\{D_a\}_{a \in V(X)}$.

(b) Show that, for a diagonal matrix D, $\Psi(D) = 0$ if and only if the diagonal of $\Psi(D)$ is equal to 0.

(c) Show that $\operatorname{rank}(\widehat{M})$ is equal to the dimension of the image under Ψ of the space of diagonal matrices.

(d) Let B commute with A, and assume B has 0 diagonal. Show that B is orthogonal to $\Psi(D)$ for all diagonal matrices D. Conclude that all matrices in $\mathrm{comm}(A)$ can be written uniquely as $\Psi(D)+B$, for one diagonal matrix D and one matrix B that commutes with A and has 0 diagonal.

(e) Let A be the adjacency matrix of a cycle C_n. What are some linearly independent elements of $\mathrm{comm}(A)$ containing non-zero diagonal entries? (think in terms of automorphisms...)

(f) Conclude that $\mathrm{rank}(\widehat{M}) = n$ if n is odd, and $\mathrm{rank}(\widehat{M}) = n/2$ if n is even (for this second part, you may use the fact that antipodal vertices are strongly cospectral, and then apply Exercise 5).