Assignment 3 - due August 18th

Send pdf with answers to dcc030ufmg@gmail.com. They must be typed up in latex.

This assignment corresponds largely to Chapter 1, to the very beginning of Chapter 4 (before section 4.1), and to Section 7.1 of the Graph Spectra and Quantum Walks book.

1st part

- Write out the definitions of cospectral vertices, parallel vertices, and strongly cospectral vertices (beginning of Chapter 4).
- Write out the definition of eigenvalue support of a vertex. Use P_3 as an example of a graph in which one of the eigenvalues is not in the eigenvalue support of a vertex.
- Read Section 7.1. Note how Lemma 7.1.1 has to do with the eigenvalues, whereas 7.1.2 and 7.1.3 are related to eigenvectors (or the projections onto eigenspaces). Note how Lemmas 7.1.2 and 7.1.3 are saying that perfect state transfer implies that the vertices are cospectral and parallel. You will see in the Exercise below that this implies the vertices are strongly cospectral, but it fact it is possible to derive a direct proof of this fact (as in the beginning of Chapter 4), and also a direct proof of Lemma 7.1.1. That is it, just read and think.
- Write out a characterization of perfect state transfer between a and b at a time τ, that is, state correctly which conditions the $E_r e_a$ and $E_r e_b$ must satisfy, along with the right condition satisfied by θ_r (no need to write the proof).

2nd part

Answer the following exercises.

Exercise 1. Using Lemma 7.1.1, show that if perfect state transfer occurs between a and b, then for all eigenvalues $\theta_r, \theta_s, \theta_k, \theta_\ell$ in their eigenvalue support, with $\theta_k \neq \theta_\ell$, we have

$$\frac{\theta_r - \theta_s}{\theta_k - \theta_\ell} \in \mathbb{Q}$$

Exercise 2. Show that two vertices a and b are strongly cospectral if, and only if, they are cospectral and parallel (make no assumption on whether perfect state transfer happens or not. One direction is indeed almost trivial.)

Exercise 3 (Read Section 1.10)**.** Assume a and b are vertices of a graph G, and let P be the permutation matrix corresponding to an automorphism π of G.

(a) Assume that there is perfect state transfer between a and b. Prove that if π fixes a, then it also fixes b. Use this to conclude that there is no perfect state transfer in a complete graph on more than 2 vertices.

(b) More weakly, assume only that a and b are strongly cospectral vertices. Nevertheless, show that if π fixes a, then it must fix b in this case as well.

(c) Show that if $Pe_a = e_b$, then a and b are cospectral.

(d) Find a graph with a pair of vertices a and b that are cospectral but not strongly cospectral.