Assignment 6 - due September 8th

Send pdf with answers to `dcc030ufmg@gmail.com`. They must be typed up in latex.

This assignment is about Chapter 7 of the book Graph Spectra and Quantum Walks, specially sections 7.4 and 7.5. However you are asked to read Section 7.3 as well.

1st part

- Read Section 7.3 until the end of the proof of Corollary 7.3.1, and read the statements of Theorem 7.3.2 and Corollary 7.3.3. Understand these things well. The remainder of this section requires some comfort with the basics of algebraic number theory. It will be very nice if you make an effort to grasp some idea of what is going on.
- Read Sections 7.4 and 7.5 carefully, specially Theorem 7.5.1.
- No need to write anything this week.

2nd part

Exercise 1. The Laplacian matrix of a graph is defined as $L = D - A$, where A is the adjacency matrix and D is a diagonal matrix containing the degrees of the vertices. First, show that 0 is eigenvalue of L, and find a corresponding eigenvector. Then show that L is positive semidefinite (to do this, consider the incidence matrix B of any orientation of the graph and show that $BB^T = L$). Finally, show that in the quantum walk model defined by L, that is, that with transition matrix $\exp(\mathrm{i}tL)$, a vertex is periodic if and only if the eigenvalues in its eigenvalue support are all integers. Further, show that if perfect state transfer occurs, having $\exp(\mathrm{i}tL)e_a = \lambda e_b$, then $\lambda = 1$.

Exercise 2. Show that perfect state transfer (adjacency matrix model) does not happen between the extremes of the path graph P_n with $n > 3$. (You may use the fact that the eigenvalues of P_n are $2\cos(k\pi/(n+1))$ for $k = 1, ..., n$). Hint: look at Corollary 7.4.2.

Exercise 3. Assume A is the adjacency matrix of a tree T, and assume A is invertible.

(a) Show (by induction, deleting a leaf?) that $\det A = \pm 1$.

(b) Show that the entries of A^{-1} are integers (you may use some nice formula for A^{-1}).

(c) Read the basics of algebraic number theory again. Show that A cannot have integer eigenvalues except for ± 1, and use the characterization of quadratic integers to show that A cannot have eigenvalues of the form $b\sqrt{\Delta}$ with $\Delta > 1$ and b integer.

(d) T is bipartite. Show (you can research) that if θ is an eigenvalue with eigenvector v, then $-\theta$ is also an eigenvalue. Relate (one of) its eigenvectors to v.

(e) Show that no eigenvalue of the form $\frac{a+b\sqrt{\Delta}}{2}$ can be in the eigenvalue support of a periodic vertex if $a \neq 0$.

(f) Conclude that the only tree with invertible adjacency matrix that contains a periodic vertex is P_2.