Assignment 1 - due August 22nd

Send answers to `dcc030ufmg@gmail.com`.

- ABOUT COLLABORATION AND/OR RESEARCH:
  - If the question has a $*$, you should not collaborate or research the question in sources other than the course notes. You can ask me questions about them though.
  - For the other questions, you can talk to your colleagues — list the names of all them (even if you only helped them) and indicate if any of them gave a decisive contribution; or research elsewhere — indicate the source.
  - Even if you collaborate or research, try to write your own solution after the discussion or after reading the source. Do not write your solutions while looking at someone else's or at a source. Do not send your written up solutions to anyone.
  - As long as you are trying to do some questions on your own, your write-ups look original and you are giving due credit, your final grade will not be affected by having done some collaboration or research.

Exercise 1 ($*$)**.** Prove that $\Theta(C_n) \geq \sqrt{n}$ if n is odd. Hint: look at the example for $n = 5$. (you need not write a formal proof)

Exercise 2. Show that if it is possible to cover the vertex set of G with $\alpha(G)$ cliques, then $\Theta(G) = \alpha(G)$.

Exercise 3 ($*$)**.** Prove that a half-space is a convex set.

Exercise 4. Using the Theorem of the Alternatives, prove Farka's lemma: $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ has no solution if and only if there is $\mathbf{y}$ so that $\mathbf{y}^T\mathbf{A} \geq \mathbf{0}$ and $\mathbf{b}^T\mathbf{y} < 0$.

Exercise 5. Prove the complementary slackness conditions, ie, if $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$ are a pair of respective optima solutions for a primal-dual pair of LPs of the form

$$(P) \quad \begin{array}{rl} \max & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{array} \qquad \Bigg| \qquad (D) \quad \begin{array}{rl} \min & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0}. \end{array}$$

then it is not possible that the ith variable in either solution is non-zero while the ith inequality in the other program is not satisfied with equality. (Hint: write extra variables $\mathbf{u}$ so that $\mathbf{Ax} + \mathbf{u} = \mathbf{b}$ and variables $\mathbf{v}$ with $\mathbf{A}^T\mathbf{y} - \mathbf{v} = \mathbf{c}$.)

Exercise 6 ($*$)**.** Compute $\chi_f(\overline{C_n})$. Prove your answer is right.

Exercise 7 ($*$)**.** Show that if $\mathbf{M}$ is positive semidefinite, then the determinant of its principal square submatrices is non-negative.

Exercise 8. Let A and B be positive semidefinite. Show that $\langle A, B\rangle = 0$ if and only if $AB = 0$.