Assignment 2 - due September 20th

Send answers to `dcc030ufmg@gmail.com`.

Exercise 1. $(*)$ Bruno's exercise: Given Euclidean spaces $\mathbb{V}_1$ and $\mathbb{V}_2$, define $\mathbb{V}_1 \oplus \mathbb{V}_2$ as the Euclidean space with elements in $\mathbb{V}_1 \times \mathbb{V}_2$ and coordinate-wise operations:

- $(\mathbf{v}_1, \mathbf{v}_2) + (\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{v}_1 + \mathbf{u}_1, \mathbf{v}_2 + \mathbf{u}_2)$.
- $\alpha(\mathbf{v}_1, \mathbf{v}_2) = (\alpha\mathbf{v}_1, \alpha\mathbf{v}_2)$.
- $\langle(\mathbf{v}_1, \mathbf{v}_2), (\mathbf{u}_1, \mathbf{u}_2)\rangle = \langle\mathbf{v}_1, \mathbf{u}_1\rangle + \langle\mathbf{v}_2, \mathbf{u}_2\rangle$.

If $\mathbb{K}_1 \subseteq \mathbb{V}_1$ and $\mathbb{K}_2 \subseteq \mathbb{V}_2$ are cones, show that

(a) $\mathbb{K}_1 \oplus \mathbb{K}_2$ is a cone.

(b) $(\mathbb{K}_1 \oplus \mathbb{K}_2)^* = \mathbb{K}_1^* \oplus \mathbb{K}_2^*$ (no need to assume anything about these cones).

(c) Decide if $\mathbb{K}_1$ and $\mathbb{K}_2$ convex and/or closed is if and only if the same holds for $\mathbb{K}_1 \oplus \mathbb{K}_2$.

Exercise 2. $(*)$ Suppose we decide to add a constraint of the form $\operatorname{rank} \mathbf{X} = 1$ to an SDP. The goal of this exercise is to show that this can leave the problem (most likely) intractable. Consider a simple loopless graph $G = (V, E)$, with adjacency matrix $\mathbf{A}$.

(a) Let $\mathcal{P} = \{\mathbf{x} \in \{0,1\}^V : \mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{0}\}$. What graph parameter is $\max\{\mathbf{1}^T\mathbf{x} : \mathbf{x} \in \mathcal{P}\}$?

Given $\mathbf{x}$, let $\hat{\mathbf{X}}$ denote

$$\hat{\mathbf{X}} = \begin{pmatrix} 1 & \mathbf{x}^T \\ & \\ \mathbf{x} & \mathbf{x}\mathbf{x}^T \\ & \end{pmatrix} \in \mathbb{S}^{\{0\}\cup V} \text{ (this means rows and columns are indexed by } \{0\} \cup V).$$

(b) Argue why, for all $\mathbf{x} \in \mathcal{P}$, we have that $\hat{\mathbf{X}} \in \mathbb{S}_+^{\{0\}\cup V}$.

(c) Let $\{\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_n\}$ be the canonical basis for $\mathbb{R}^{\{0\}\cup V}$. Argue why, for all $\mathbf{x} \in \mathcal{P}$, we have that $\hat{\mathbf{X}}\mathbf{e}_0 = \operatorname{diag} \hat{\mathbf{X}}$.

(d) Argue why, for all $\mathbf{x} \in \mathcal{P}$, we have $\hat{\mathbf{X}}_{ab} = 0$ for all $ab \in E(G)$.

(e) In the following program, we will optimize over matrices in $\mathbb{S}^{\{0\}\cup V}$ that satisfy the properties you showed above that $\hat{\mathbf{X}}$ satisfy for when $\mathbf{x} \in \mathcal{P}$ (but that are not necessarily defined as such).

$$\begin{aligned} \max \quad & \sum_{a\in V} \mathbf{X}_{0a} \\ \text{subject to} \quad & \mathbf{X}_{00} = 1, \\ & \mathbf{X}\mathbf{e}_0 = \operatorname{diag} \mathbf{X}, \\ & \mathbf{X}_{ab} = 0 \text{ for all } ab \in E(G), \\ & \mathbf{X} \in \mathbb{S}_+^{\{0\}\cup V}. \end{aligned}$$

Explain why the optimum of this SDP might still not be equal to the parameter you found in (a) (you could give a shady reason, or provide an example and make me happy.)

(f) Argue why adding the constraint rank $\mathbf{X} = 1$ to program above will make its optimum value equal to the parameter you found in (a). Conclude that solving an SDP with a rank $= 1$ constraint is NP-hard.

(g) For a bonus, show that the optimum of the SDP in (e) is $\vartheta(G)$ (or at least prove one side of an inequality).

Exercise 3. $(*)$ Show that the Lorentz cone is self dual.

Exercise 4. Let $p(x)$ be a polynomial of degree d, real coefficients. We would like to find out if $p(x) \geq 0$ for all x. This would happen if there are polynomials $q_1(x), ..., q_m(x)$ so that

$$p(x) = \sum_{i=1}^{m} q_i(x)^2.$$

(a) Write this decision problem as the problem of finding a feasible solution to an SDP. (Hint: let $[x]_d$ be the "vector" whose kth entry is the monomial x^{k-1}. Note that $q(x)^2 = [x]_d^T \mathbf{q}\mathbf{q}^T [x]_d$ where $\mathbf{q}$ is the vector with coefficients of $q(x)$).

(b) Every real polynomial can be factored over $\mathbb{R}$ as polynomials with real coefficients which are either linear or quadratic. Using this, show that $p(x) \geq 0$ for all $x \in \mathbb{R}$ if and only if only it can be written as a sum of squares. (Hint: could linear terms appear?)

(c) Let $p(x)$. Express the global minimum $\min\{p(y) : y \in \mathbb{R}\}$ as the optimum of an SDP.

(d) For a bonus point, formulate the problem of deciding if $p(x_1, ..., x_n)$ and total degree d can be written as the sum of squares of polynomials in n variables as the feasibility of an SDP.

Exercise 5. Prove Vinicius's exercise: for any cone, $\mathbb{K}^{**}$ is the closure of the convex hull of $\mathbb{K}$ (ie, show that $\mathbb{K}^{**}$ is the smallest closed and convex cone that contains $\mathbb{K}$). Hint: hyperplane separation?

Exercise 6. Assume $\mathbb{K}$ is a closed, convex cone. From the exercise above, we know that $\mathbb{K}^{**} = \mathbb{K}$. Here, use this fact to prove that

(a) If K is pointed, then $\mathbb{K}^*$ has non-empty interior.

(b) If K has non-empty interior, then $\mathbb{K}^*$ is pointed.

(Hint: use that the existence of an interior point implies that the cone contains a basis of the space.)

Exercise 7. Given $\mathbf{M}$ symmetric, with eigenvalues $\lambda_1 \geq ... \geq \lambda_n$, show that $\lambda_1 + ... + \lambda_k$ is equal to

$$\begin{aligned} \max \quad & \langle \mathbf{M}, \mathbf{X} \rangle \\ \text{subject to} \quad & \langle \mathbf{I}, \mathbf{X} \rangle = k \\ & \mathbf{X} \preccurlyeq \mathbf{I} \\ & \mathbf{X} \succcurlyeq \mathbf{0}. \end{aligned}$$