Assignment 3 - due October 25th

Send answers to `dcc030ufmg@gmail.com`.

Exercise 1. ($*$) Consider the maxcut formulation

$$\begin{array}{rl} \max & \left\langle \frac{1}{4}\mathbf{L}, \mathbf{Y} \right\rangle \\ \text{subject to} & \operatorname{diag} \mathbf{Y} = \mathbf{1} \\ & \mathbf{Y} \succcurlyeq \mathbf{0}. \end{array}$$

Write the dual formulation of this formulation. Then, verify that $\mathbf{I}$ is a Slater point for the primal, and find a Slater point for the dual.

Exercise 2. ($*$) Show that

$$\texttt{STAB}(G) = \operatorname{conv}\left(\texttt{FRAC}(G) \cap \{0,1\}^V\right).$$

Exercise 3. ($*$) Show that if G is bipartite, then

$$\texttt{STAB}(G) = \texttt{QSTAB}(G) = \texttt{FRAC}(G).$$

Exercise 4. ($*$) Let G be a graph, $\mathbf{x} \in \mathbb{R}^V$. Show that $\mathbf{x} \in \texttt{TH}(G)$ if, and only if, there is orthonormal representation ρ of $\overline{G}$ with $\mathbf{x}_i = \langle \rho_0, \rho_i \rangle^2$ for all $i \in V$. (You can use results proved in class).

Exercise 5. The max weight bissection problem asks for a cut $\delta(S)$ of maximum weight requiring that $|S| = n/2$.

(a) Come up with an equality that should be added to the ± 1 formulation of maxcut that does the deed.

(b) Come up with a strong equality of the form $\langle ?, \mathbf{X} \rangle =?$ to be added to the SDP relaxation.

(c) Write its dual.

Exercise 6. The max k-cut problem asks for a partition of V into k parts that maximizes the weights of the edges between them. Instead of assigning values ± 1 to each vertex (two possible values in $\mathbb{R}^1$), one could assign one of k possible vectors $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$, in $\mathbb{R}^{k-1}$, so that $\mathbf{v}_i^T \mathbf{v}_j = -1/(k-1)$ if $i \neq j$.

(a) Prove that such vectors exist, for any k.

(b) Write the analogous of the ± 1 formulation.

(c) Write the SDP relaxation.

(d) Try to guess how to use a solution of the relaxation to find the partition into k parts.

Exercise 7. Let G be a graph. A distance 1 embedding of G onto $\mathbb{R}^d$ is a map $\rho : V(G) \to \mathbb{R}^d$ so that, for all $ij \in E$, we have

$$||\rho_i - \rho_j||_2 = 1.$$

Show that the optimum of the SDP below is finite and is attained (meaning, find Slater points to itself and its dual), and is equal to the square of the radius of the smallest ball containing a distance 1 embedding of G.

$$\begin{array}{rl} \min & t \\ \text{subject to} & \operatorname{diag} \mathbf{X} - t\mathbf{1} \leq \mathbf{0} \\ & \mathbf{X}_{ii} - 2\mathbf{X}_{ij} + \mathbf{X}_{jj} = 1 \quad \text{for all } ij \in E \\ & \mathbf{X} \in \mathbb{S}_+^V. \end{array}$$

Hint: how to go from psd matrices to vectors?

Exercise 8. Prove that

$$\vartheta(G) = \min\{\lambda : \mathbf{Z}_{00} = \lambda,\ \mathbf{Z}_{0i} = \mathbf{Z}_{ii} = 1\ \forall\ i \in V(G),\ \mathbf{Z}_{ij} = 0 \text{ if } ij \in \overline{E},\ \mathbf{Z} \in \mathbb{S}_+^{\{0\} \cup V(G)}\}.$$

Further, find a way of turning this formulation into yet another equivalent definition of the theta function $\vartheta(G; \mathbf{w})$.

Exercise 9. Prove that, if $\mathbb{K}$ and $\mathbb{L}$ are closed and convex cones, then

$$(\mathbb{K} \cap \mathbb{L})^* = \mathbb{K}^* + \mathbb{L}^*.$$