Assignment 4 - due December 6th

Send answers to `dcc030ufmg@gmail.com`.

Exercise 1. We want to use Lasserre hierarchy to approximate $\mathtt{STAB}(G)$. For that, we will use polynomials $p_{ab}(x) = 1 - x_a - x_b \in \mathbb{R}[x_v : v \in V(G)]$, for all $ab \in E(G)$. Of course, having $p_{ab}(x) \geq 0$ for all $ab \in E(G)$ is equivalent to defining $\mathtt{FRAC}(G)$, which is where we start.

Given $f : 2^V \to \mathbb{R}$, let $\mathbf{M}_f^t$ and $\mathbf{M}_{p_{ab}*f}^t$ be the truncations of $\mathbf{M}_f$ and $\mathbf{M}_{p_{ab}*f}$ to the subsets of at most t elements.

Show that the following are all equivalent, for $t \geq 1$.

(a) $\mathbf{M}_f^{t+1} \succcurlyeq \mathbf{0}$, and $\mathbf{M}_{p_{ab}*f}^t \succcurlyeq \mathbf{0}$ for all $ab \in E(G)$.

(b) $\mathbf{M}_f^{t+1} \succcurlyeq \mathbf{0}$, sand $f(\{a,b\}) = 0$ for all $ab \in E(G)$.

(c) $\mathbf{M}_f^{t+1} \succcurlyeq \mathbf{0}$, and $f(U) = 0$ for all $U \subseteq V(G)$ which is not independent, with $|U| \leq 2t+2$.

Start by noting that $\mathbf{M}_f^{t+1} \succcurlyeq \mathbf{0}$ implies $f(U) \geq 0$ for all U subset of at most $2t+2$ elements.

Exercise 2. Show that

$$\max\{\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} : \mathbf{1}^{\mathsf{T}}\mathbf{x} = 1,\ \mathbf{x} \geq \mathbf{0}\} = \frac{1}{2}\left(1 - \frac{1}{\omega(G)}\right).$$

Exercise 3. Recall the definition of $\mathbf{A}_i$ as the 01-matrix with rows and columns indexed by strings of n 0s and 1s, and an entry equal to 1 if and only if the corresponding strings differ in exactly i coordinates. Show that there are constants c_i and b_i so that

$$\mathbf{A}_1\mathbf{A}_i = c_i\mathbf{A}_{i-1} + b_i\mathbf{A}_{i+1}.$$

Exercise 4. Recall the definition of $\Omega(d)$ as the graph with the unit vectors of $\mathbb{R}^d$ as its vertices, and two vertices adjacent if and only if they are orthogonal. Recall that $\xi(G)$ is the least d so that there is a homomorphism from G to $\Omega(d)$. The graph $S(d,\alpha)$ has the same set of vertices, and two adjacent if and only if their inner product is $\leq \alpha$. If G has a homomorphism to $S(d,\alpha)$, we say G has a vector $(1-(1/\alpha))$ colouring.

Prove that the following conditions are all equivalent.

1. G is bipartite.
2. $\xi(G) \leq 2$.
3. G has a vector 2-colouring.

Exercise 5 (?)**.** Find a feasible solution to

$$\begin{aligned} \max \quad & \langle \mathbf{J}, \mathbf{X} \rangle \\ \text{subject to} \quad & \mathrm{X}_{ij} + \mathrm{Z}_{ij} = 0 \ \forall\ ij \in E(G) \\ & \mathrm{tr}(\mathbf{X} + \mathbf{Z}) = 1 \\ & \mathbf{X} \in \mathbb{S}_+^n,\ \mathbf{Z} \in \mathbb{S}_{\text{copo}}^n \end{aligned}$$

with objective value equal to $\chi_f(\overline{G})$.