CS:4330 Theory of Computation

Spring 2018

Regular Languages

Equivalences between Finite automata and REs

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Readings for this lecture

Chapter 1 of [Sipser 1996], 3rd edition. Section 1.3.

Finite automata and regular expressions are equivalent

Theorem

A language is regular if and only if some regular expression describes it.

Proof ideas

1. If a language A is described by a regular expression R then A is recognized by an NFA, therefore A is regular

   There is an NFA N such that N recognizes $\mathscr{L}(R)$

2. If a language A is regular, it means that it is recognized by a DFA. Then we can always deduce a regular expression from it.

   Turn DFA into equivalent regular expression

Part 1: From regular expressions to NFAs

By induction on the length of R:

- ▷ Base cases (R has length 1):
  - $R = \{\mathrm{a}\}$
  - $R = \epsilon$
  - $R = \varnothing$

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- Inductive case: let R have length $k > 1$. Assume that for any smaller regular expression, there is an NFA.
  R may be one of the following cases:
  - $R = R_1 \cup R_2$
  - $R = R_1 R_2$
  - $R = (R_1)^*$

Part 2: From DFAs to regular expressions

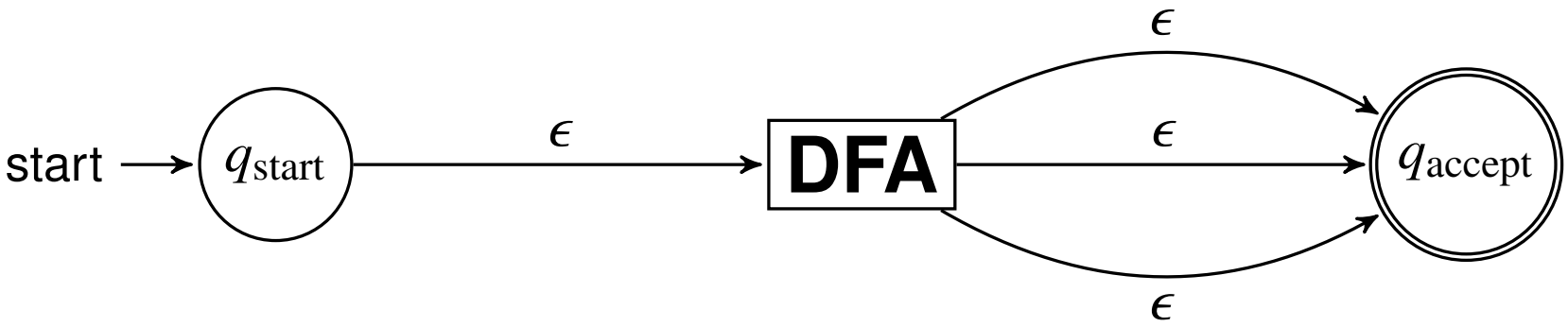
1. Define Generalized Nondeterministic Finite Automaton (GNFA in short).
   Instead of $\delta : Q \times \Sigma \to Q$, we use $\delta : Q \times RE \to Q$
   - Arrows labelled with regular expressions
   - Blocks of symbols instead of one symbol at a time
   - One start and one accept state

2. How to convert any DFA to an equivalent GNFA

3. Algorithm to convert any GNFA to an equivalent GNFA with 2 states

4. Convert a 2-state GNFA to an equivalent RE.

Step 1: DFA to GNFA



- ▷ Add unique and distinct start and accept states
- ▷ Edges with multiple labels become regexp labels
- ▷ If internal states (q_1, q_2) don't have an edge between them, add one labeled with $\varnothing$
  - ▶ This should be such that q_{start} has no incoming edges and q_{accept} has no outgoing edges.

Step 2: Eliminate states from GNFA

While machine has more than 2 states:

- ▷ Pick and internal state, rip it out
- ▷ Re-label the arrows with regular expressions to account for the missing state

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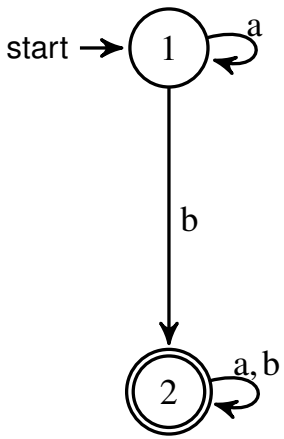

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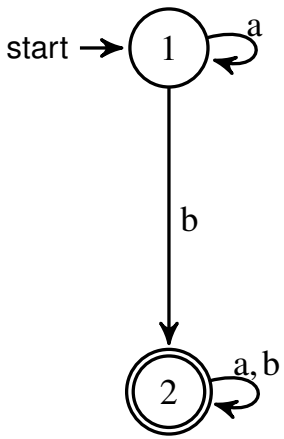

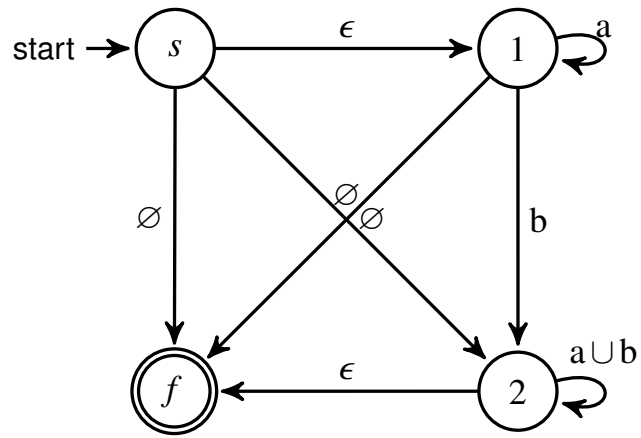

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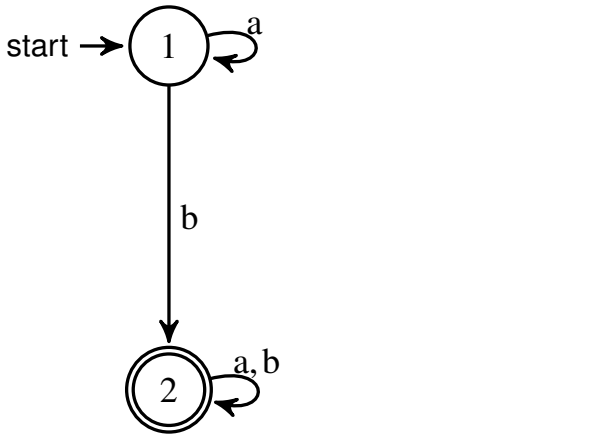

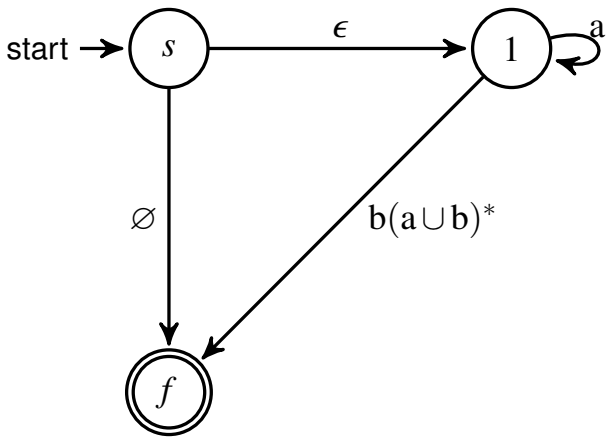

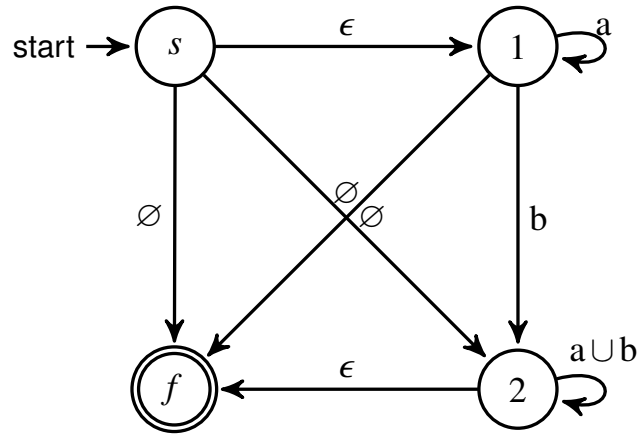

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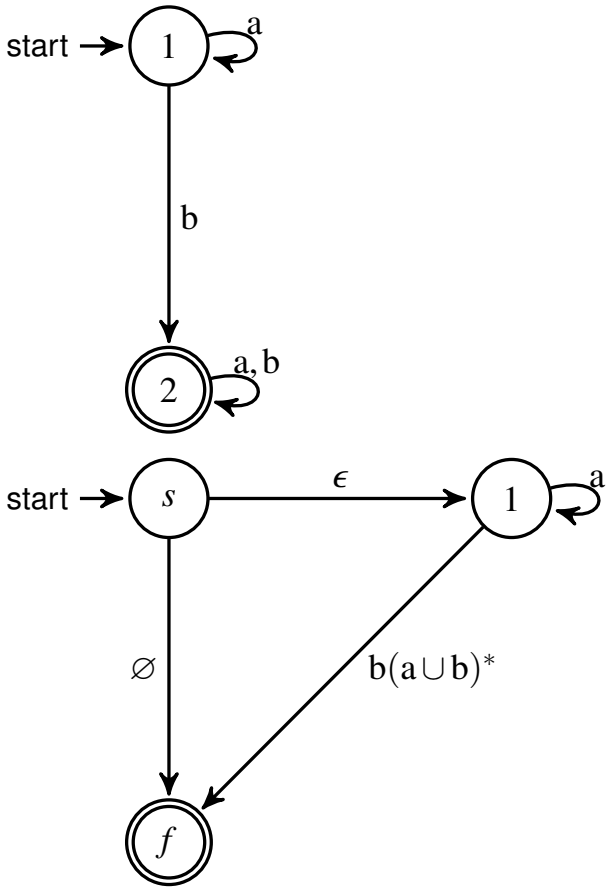
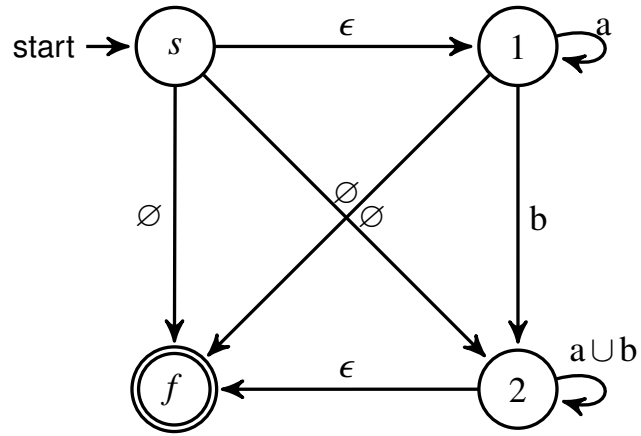
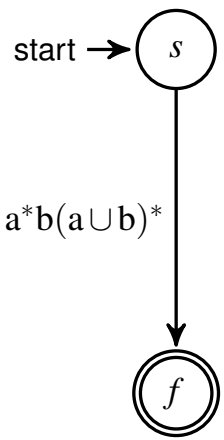






GNFA: definition and acceptance

A GNFA is a tuple $(Q, \Sigma, \delta, q_{\text{start}}, q_{\text{accept}})$

- Q is the set of states, Σ is the finite alphabet (not regexps)
- q_{start}: initial state (unique, no incoming edges)
- q_{accept}: accepting state (unique, no outgoing edges)
- $\delta : (Q \setminus \{q_{\text{accept}}\}) \times (Q \setminus \{q_{\text{start}}\}) \to R$
  - R is the set of all regexps over Σ

A GNFA accepts a string $w \in \Sigma^*$ if $w = w_1, \ldots, w_k$, with each $w_i \in \Sigma^*$ and a sequence of states $q_0, \ldots, q_k$ exists such that:

- $q_0 = q_{\text{start}}$ is the start state
- $q_k = q_{\text{accept}}$ is the accept state
- for each i, we have $w_i \in \mathscr{L}(R_i)$, where $R_i = \delta(q_{i-1}, q_i)$, i.e. R_i is the expression on the arrow from q_{i-1} to q_i

CONVERT

Given a DFA M, let G be its GNFA. CONVERT(G) yields the equivalent regexp.

1. Let k be the number of states of G
2. If $k = 2$, then G, return the regexp labeling its single transition
3. Select any state $q_{\text{rip}} \in Q \setminus \{q_{\text{start}}, q_{\text{accept}}\}$ and let G' be the GNFA $(Q', \Sigma, \delta', q_{\text{start}, q_{\text{accept}}})$ such that

   $$Q' = Q \setminus \{q_{\text{rip}}\}$$

   and for any $q_i \in Q' \setminus \{q_{\text{accept}}\}$ and any $q_j \in Q' \setminus \{q_{\text{start}}\}$, let

   $$\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) \cup (R_4)$$

   for $R_1 = \delta(q_i, q_{\text{rip}})$, $R_2 = \delta(q_{\text{rip}}, q_{\text{rip}})$, $R_3 = \delta(q_{\text{rip}}, q_j)$ and $R_4 = \delta(q_i, q_j)$
4. Return CONVERT(G')

Is CONVERT correct?

Theorem

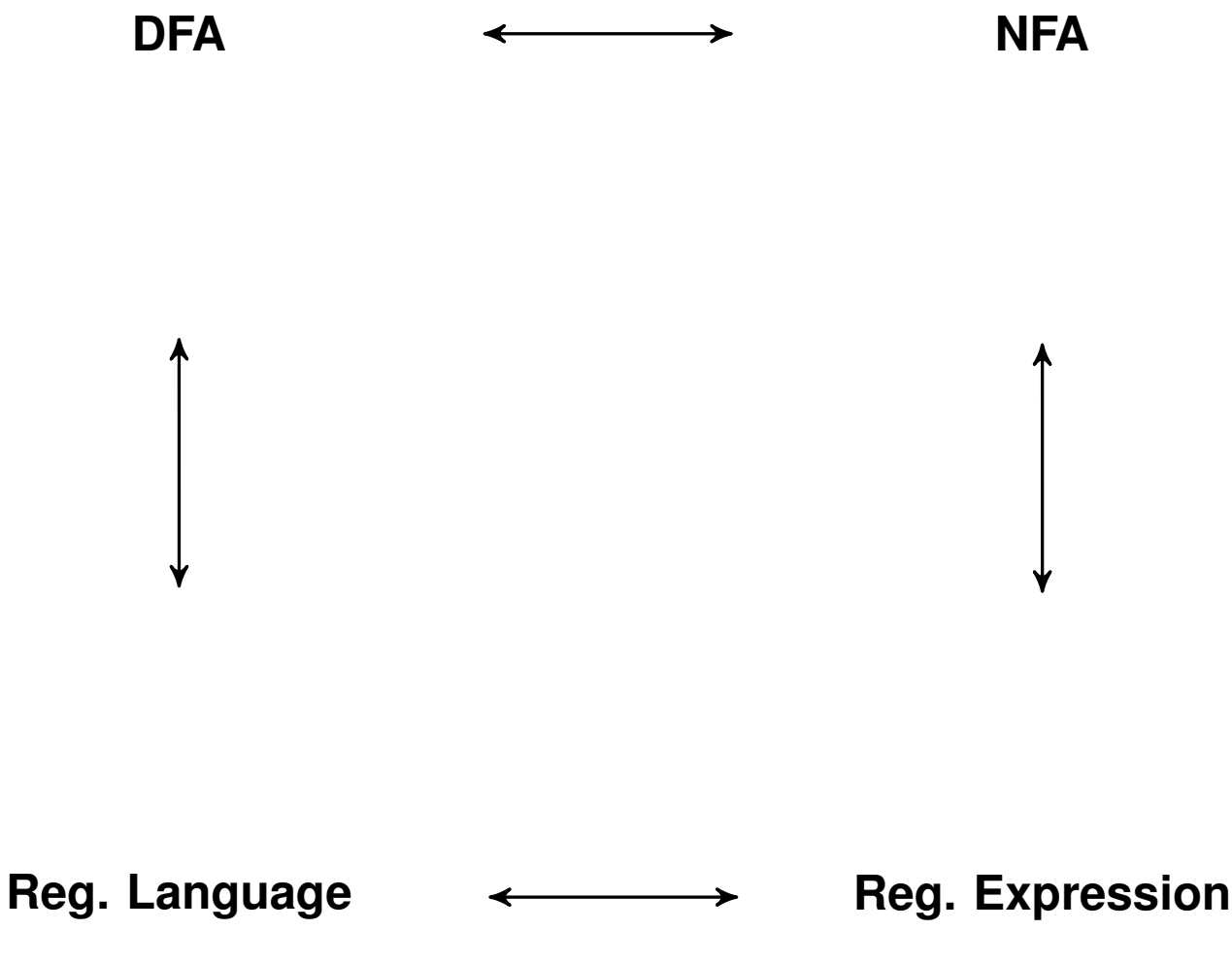
Given any GNFA G, CONVERT(G) *is equivalent to G.*

Proof idea

By induction on k, the number of states of G.

- ▷ Base step: $k = 2$
  Show that the regexp labeling its single arrow describe all accepting strings of G
- ▷ Inductive step: assume it holds for $k - 1$. Show that G and G' are equivalent (i.e. accept the same words), then by the induction hypothesis so it will be for CONVERT(G').

The Complete Picture



Limits of finite automata

Are the following languages regular?

- $L_1 = \{w \mid w \text{ has equal number of 1s and 0s}\}$

- $L_2 = \{w \mid w \text{ has equal number of occurrences of 01 and 10}\}$