CS:4330 Theory of Computation
Spring 2018

Computability Theory

Other *PSPACE-complete* problems

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Readings for this lecture

Chapter 8 of [Sipser 1996], 3rd edition. Section 8.3.

A Formula Game

- Consider a quantified Boolean formula $\varphi = \exists x_1 \forall x_2 \exists x_3 \ldots Q x_m . \psi$, with $Q \in \{\exists, \forall\}$, in prenex normal form
- The game consists of moves by a player E and a player A which take turns assigning values to the variables x_i
- Player A assigns values to the $\forall$ bound variables and player E to the $\exists$ bound variables
- The order of play is the same as in the quantifier prefix
- If ψ is true at the end of the game, E wins, otherwise A wins
- A player is said to have a *winning strategy* if the player can always win the game by making the right moves, no matter how the other player moves

Example of Formula Game

Consider the formula

$$\varphi = \exists x_1 \forall x_2 \exists x_3 . \left[(x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (\neg x_2 \vee \neg x_3) \right]$$

- ▷ In the formula game for φ, player E picks the value for x_1, then player A for x_2 and finally E picks the value for x_3
- ▷ If E selects $x_1 = 1$, thus E has the winning strategy by selecting to x_3 the negation of whatever A picks to x_2
- ▷ If

  $$\varphi = \exists x_1 \forall x_2 \exists x_3 . \left[(x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (x_2 \vee \neg x_3) \right]$$

  then A always wins because no matter how E assigns x_1, A may select $x_2 = 0$, which falsifies the last two clauses independently of the value of x_3

$\mathit{FORMULA\text{-}GAME}$ is *PSPACE-complete*

Theorem

The problem of determining which player has a winning strategy in a formula-game associated with a particular formula is PSPACE-complete, i.e.

$$\mathit{FORMULA\text{-}GAME} = \left\{ \varphi \;\middle|\; \begin{array}{l} \text{Player } E \text{ has a winning strategy in the} \\ \text{formula game } \varphi \end{array} \right\}$$

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Proof idea

Simply show that $FORMULA\text{-}GAME = TQBF$.

- ▷ A formula is true exactly when Player E has a winning strategy in the associated formula game.

Generalized Geography

This is a child game where players take turns naming cities from anywhere in the world. Each city chosen must begin with the same letter that ended the previous city name and no duplication is allowed.

Graph model

A directed graph $G = (V, E)$ whose nodes are cities of the world and an arrow goes from node n_1 to node n_2 if the city labeling n_2 starts with the letter that ends the name labeling node n_1.

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$$GG = \left\{ \langle G, b \rangle \;\middle|\; \begin{array}{l} \text{Player 1 has a winning strategy for the generalized geography} \\ \text{game played on graph } G \text{ starting at node } b \end{array} \right\}$$

Theorem

GG is PSPACE-complete.

GG is PSPACE

The following algorithm M decides whether player 1 has a winning strategy for game *GG* game:

$M =$"On input $\langle G, b \rangle$ where G is a directed graph and b is a node of G:

1. If b has outdegree 0, *reject*, player 1 loses immediately
2. Remove node b and all connected arrows to get G'
3. For each node $b_1, \ldots, b_k$ that b originally pointed, recursively call M on $\langle G', b_i \rangle$
4. If one of $\langle G', b_i \rangle$ returns "reject", player 1 would choose b_i, and M *accepts*.
5. If all of these $\langle G', b_i \rangle$ return "accept", player 2 has a winning strategy in the original game, so M returns *reject*."

GG is PSPACE-hard

Theorem

FORMULA-GAME $\leq_P$ *GG*.

- Let φ be a formula game, where the body of φ is a CNF containing m clauses. We construct a graph $G = (V, E)$ and a special node b such that player E has a winning strategy in the formula game φ iff player 1 has a winning strategy in (G, b) in *GG*.
- $V = V_1 \cup V_2$, where
  - $V_1 = \{b_i, x_i, \bar{x}_i, e_i \mid 1 \leq i \leq k, k = |X|\}$ (assume $|X|$ is odd), and
  - $V_2 = \{c, c_j, c_{j,i} \mid 1 \leq j \leq k, c_j = (c_{j,1} \vee \cdots \vee c_{j,k})\}$
- $E = E_1 \cup E_2 \cup E_3$, where
  - E_1 constructs a chain of "diamonds" among V_1
  - E_2 constructs a tree among V_2
  - E_3 connects V_1 and V_2
- $b = b_1$