$$\#(x1) = 2(\#x) + 1,$$

or in short,

$$\#(xc) = 2(\#x) + c \tag{4.4}$$

for $c \in \{0,1\}$. From the machine above, we see that for any state $q \in \{\mathbf{0},\mathbf{1},\mathbf{2}\}$ and input symbol $c \in \{0,1\}$,

$$\delta(q,c) = (2q + c) \bmod 3. \tag{4.5}$$

This can be verified by checking all six cases corresponding to possible choices of q and c. (In fact, (4.5) would have been a great way to *define* the transition function formally—then we wouldn't have had to prove it!) Now we use the inductive definition of $\widehat{\delta}$ to show (4.3) by induction on $|x|$.

Basis

For $x = \epsilon$,

$$\begin{aligned}
\widehat{\delta}(\mathbf{0},\epsilon) &= \mathbf{0} && \text{by definition of } \widehat{\delta} \\
&= \#\epsilon && \text{since } \#\epsilon = 0 \\
&= \#\epsilon \bmod 3.
\end{aligned}$$

Induction step

Assuming that (4.3) is true for $x \in \{0,1\}^*$, we show that it is true for xc, where $c \in \{0,1\}$.

$$\begin{aligned}
\widehat{\delta}(\mathbf{0},xc) &= \delta(\widehat{\delta}(\mathbf{0},x),c) && \text{definition of } \widehat{\delta} \\
&= \delta(\#x \bmod 3, c) && \text{induction hypothesis} \\
&= (2(\#x \bmod 3) + c) \bmod 3 && \text{by (4.5)} \\
&= (2(\#x) + c) \bmod 3 && \text{elementary number theory} \\
&= \#xc \bmod 3 && \text{by (4.4).}
\end{aligned}$$

Note that each step has its reason. We used the definition of δ, which is specific to this automaton; the definition of $\widehat{\delta}$ from δ, which is the same for all automata; and elementary properties of numbers and strings.

Some Closure Properties of Regular Sets

For $A, B \subseteq \Sigma^*$, recall the following definitions:

$$\begin{aligned}
A \cup B &= \{x \mid x \in A \text{ or } x \in B\} && \text{union} \\
A \cap B &= \{x \mid x \in A \text{ and } x \in B\} && \text{intersection} \\
\sim A &= \{x \in \Sigma^* \mid x \notin A\} && \text{complement}
\end{aligned}$$