P_3-Hull number of graphs with diameter two

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Joint work with Márcia R. Cappelle, Hebert Coelho, Fábio Protti, Braully R. Silva and Uéverton S.Souza

Motivation

The spread of disease on a square grid [Bollobás (2006)].

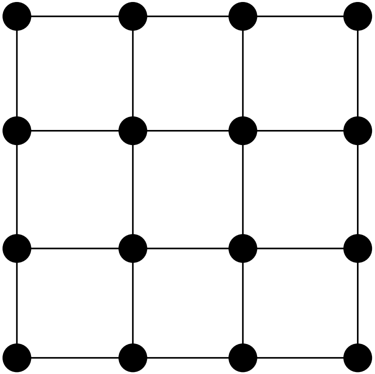

Figura: 4×4 Grid.

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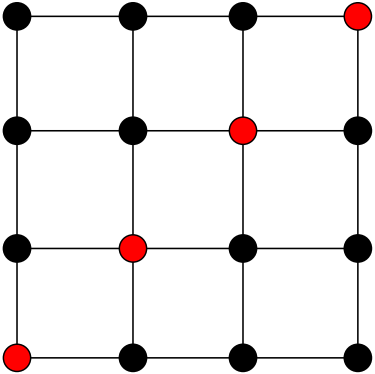

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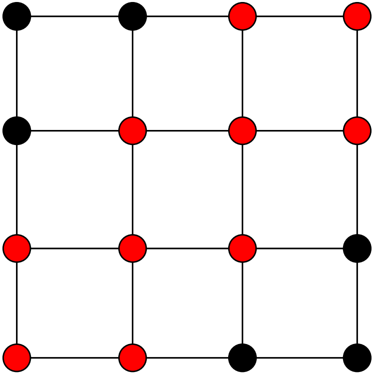

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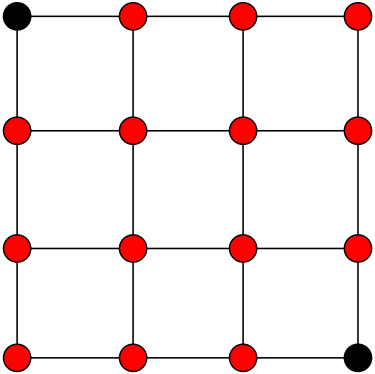

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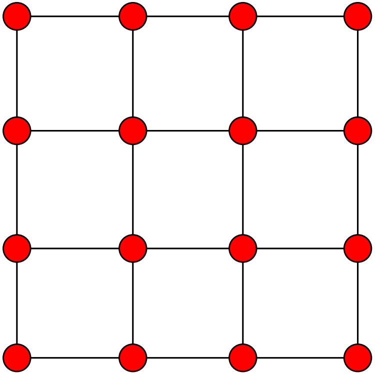

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Introduction

Graph Convexity

Consider finite, simple and undirected graphs. Let G be a graph with vertex set $V(G)$. A *graph convexity* on $V(G)$ is a collection $\mathcal{C}$ of subsets of $V(G)$ such that:

- $\emptyset, V(G) \in \mathcal{C}$;
- $\mathcal{C}$ is closed under intersections.

The sets in $\mathcal{C}$ are called *convex sets*.

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The P_3-convexity on a graph G

Let G be such a graph and a set $S \subseteq V(G)$:

- Define the P_3-*interval* $I[S]$ as the set S with the set of vertices in $V(G) \setminus S$ with at least two neighbors in S.
- If $I[S] = S$, then the set S is P_3-*convex*.
- The P_3-*convex hull* $H(S)$ of S is the smallest P_3-convex set containing S.
- If $H(S) = V(G)$ we say that S is a P_3-*hull set* of G.
- The cardinality $h(G)$ of a minimum P_3-hull set in G is called the P_3-*hull number* of G.

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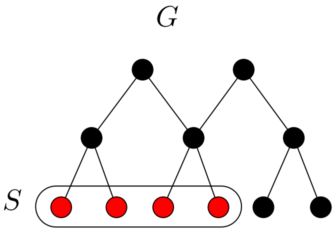

Figura: A set $S \subseteq V(G)$.

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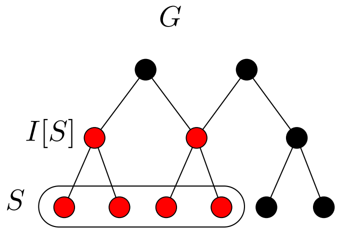

Figura: The P_3-interval $I[S]$ of S.

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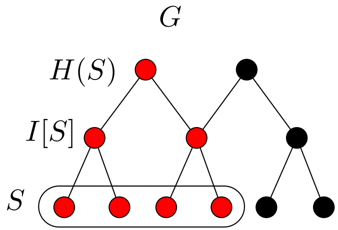

Figura: The P_3-convex hull $H(S)$ of S.

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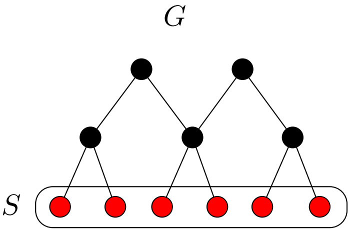

Figura: Another set $S \subseteq V(G)$.

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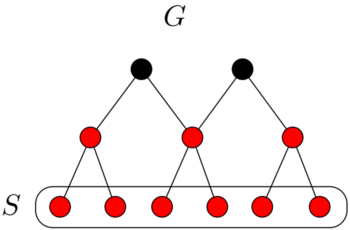

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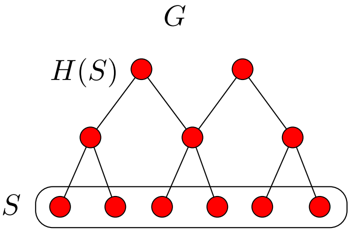

Figura: The P_3-convex hull $H(S) = V(G)$. $h(G) = 6$.

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- Some convexities in graphs are defined by a set P of paths in graphs
  - *Geodetic convexity*: shortest paths;
  - *Monophonic convexity*: induced paths;
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  - P_3-convexity: paths with three vertices.

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Geodetic convexity

- NP-hard for bipartite graphs [Araujo et al. (2013)].

Characterizations, polynomial time algorithms and bounds

- Cographs and split graphs [Dourado et al. (2009)]
- $(q, q-4)$-graphs [Araujo et al. (2013)]
- $\{paw, P5\}$-free graphs [Dourado et al. (2016)]
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Monophonic and triangle path convexity

The hull number can be determined in polynomial time for general graphs.

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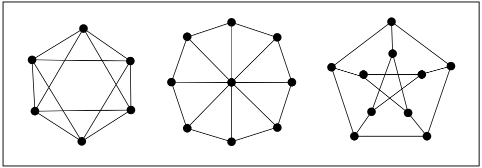

Co-convex set

Definition

A set T of vertices is *co-convex* if every vertex of T has at most one neighbor outside T

Fact

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Diameter two graph with a cut-vertex

Proposition

Let G be a diameter two graph with a cut-vertex v. Then, $h(G) = \omega(G - v)$, where $\omega(G)$ is the number of connected components of G.

Proof idea

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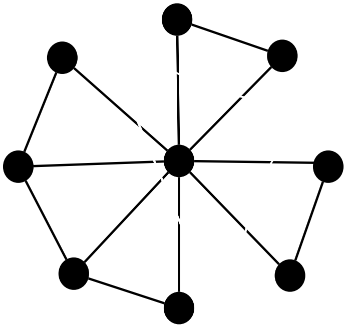

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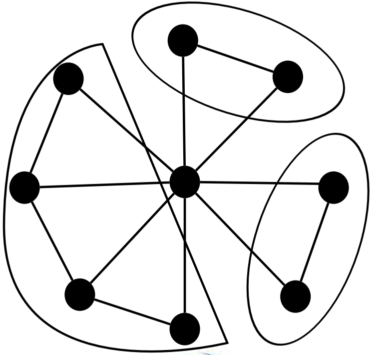

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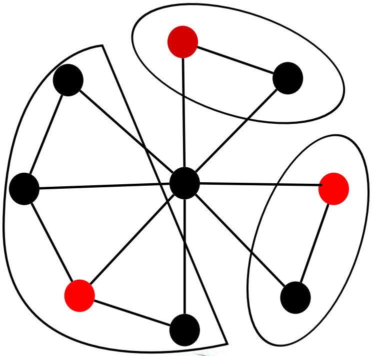

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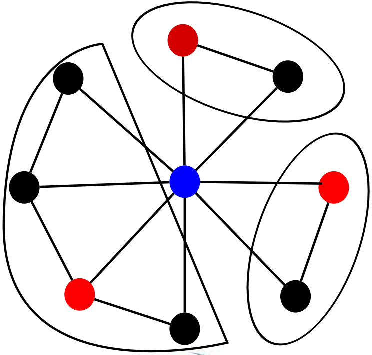

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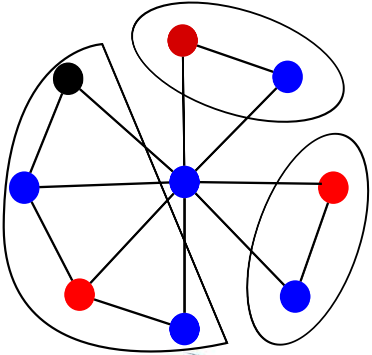

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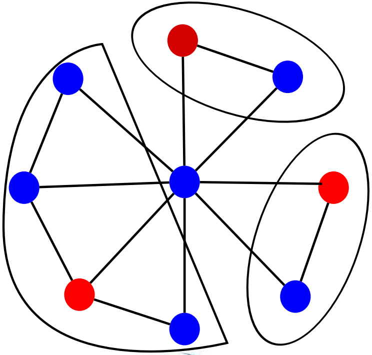

Hull set and dominating sets

Lemma

Let G be a biconnected diameter two graph and $S' \subseteq V(G)$. If $H(S')$ is a dominating set of G, then for any $v \in V(G) - H(S')$ the set $S = S' \cup \{v\}$ is a hull set of G.

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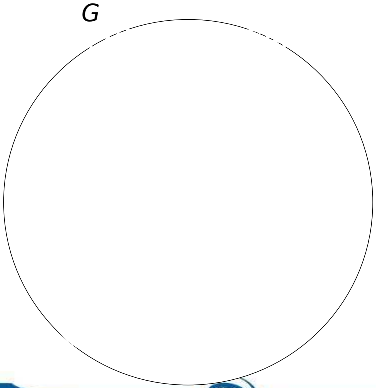

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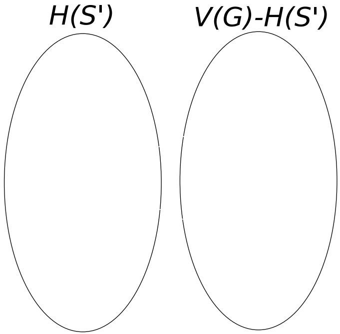

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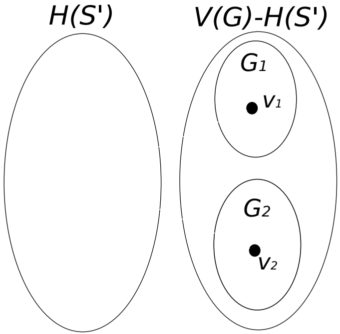

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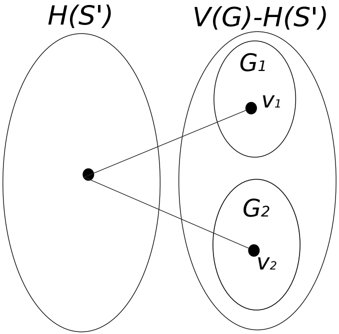

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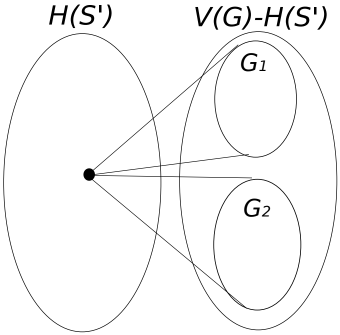

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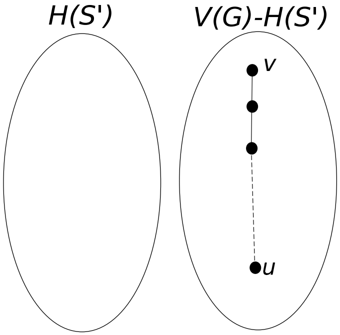

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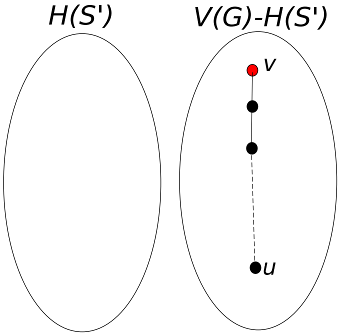

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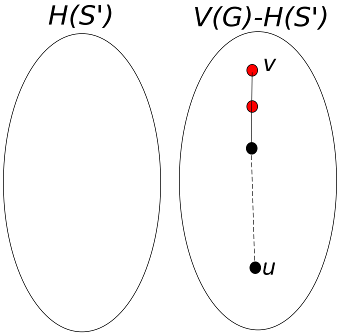

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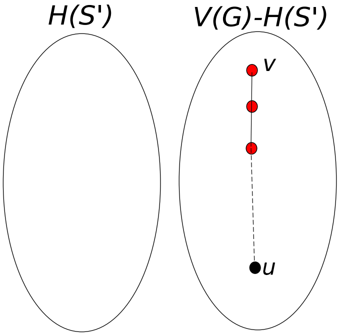

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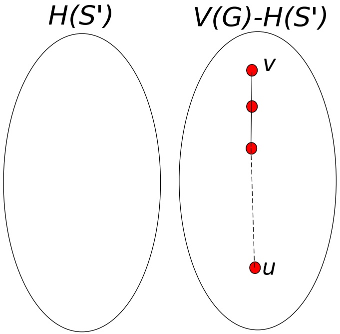
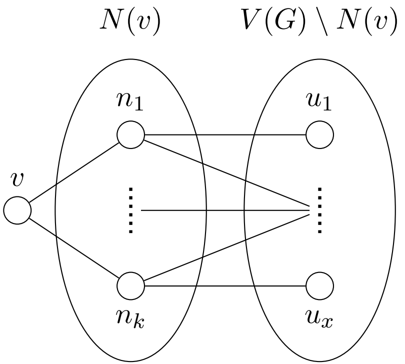


Hull number and dominating sets



Corollary

If G is a biconnected diameter two graph, then $h(G) \leq \delta + 1$.

Hull number and cut of vertices

Corollary

If G is a biconnected diameter two graph, then $h(G) \leq c(G) + 1$, where $c(G)$ is a cardinality of a minimum cut of vertices in G.

Properties of diameter two graphs

Any set C can partition $V(G)$

- $N = N(C) - C$
- $O = N[N] - N[C]$

Proposition

If G is a diameter two graph, then $V(G) = C \cup N \cup O$.

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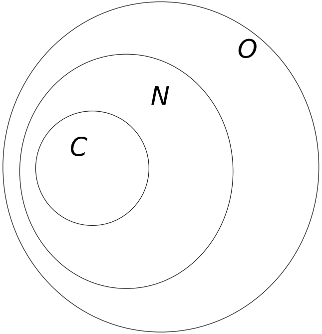

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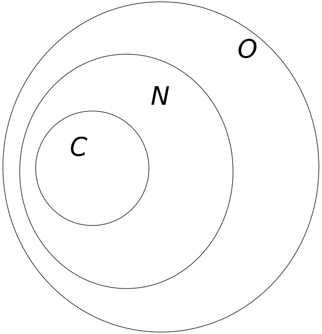

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- $O = N[N] - N[C]$

Proposition

If G is a diameter two graph, then $V(G) = C \cup N \cup O$.

Properties of diameter two graphs

Proposition

Let G be a diameter two graph and $S \subseteq V(G)$. Consider $C = H(S)$, $N = N(C) - C$ and $O = V(G) - N[C]$. Then, the following statements are true:

a) *If $O = \emptyset$, then $H(S)$ is a dominating set.*

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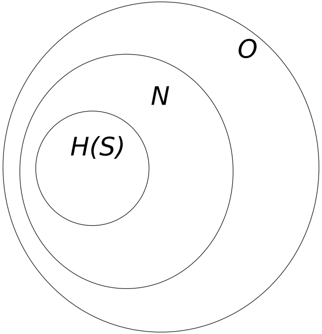

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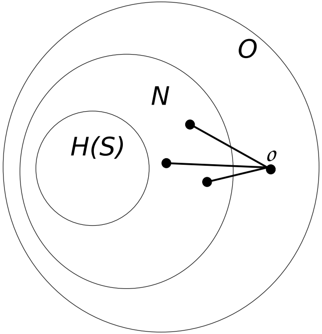

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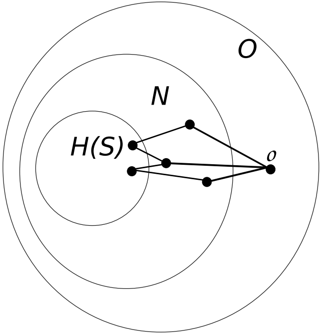

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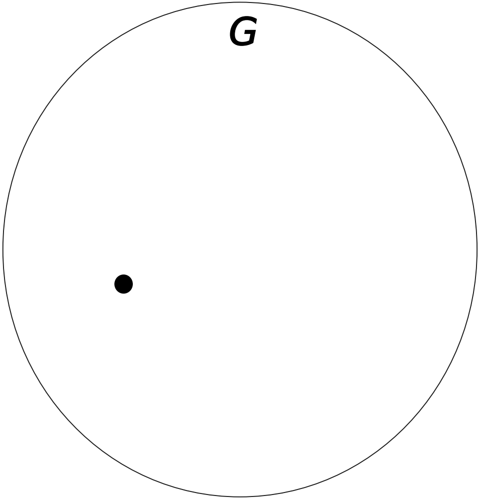

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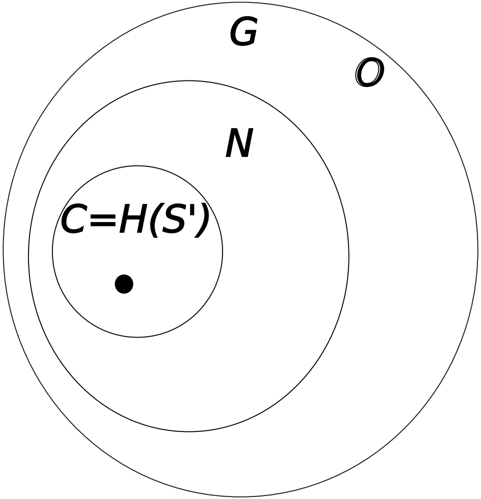

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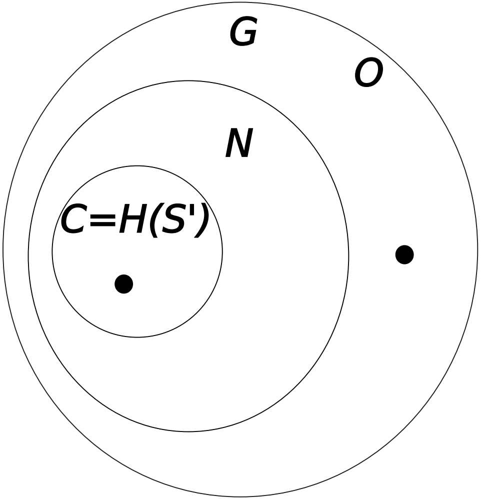

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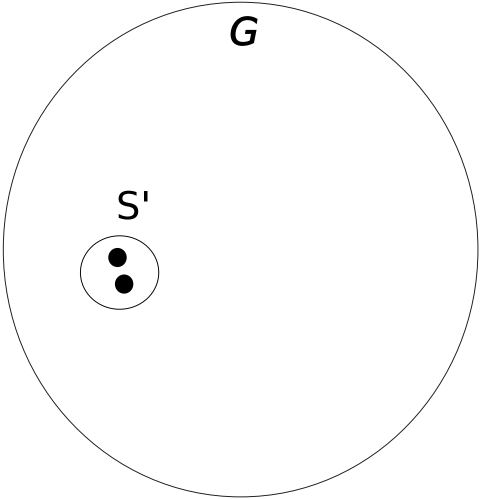

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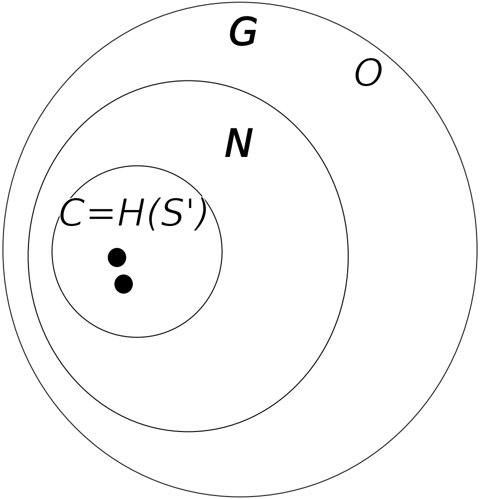

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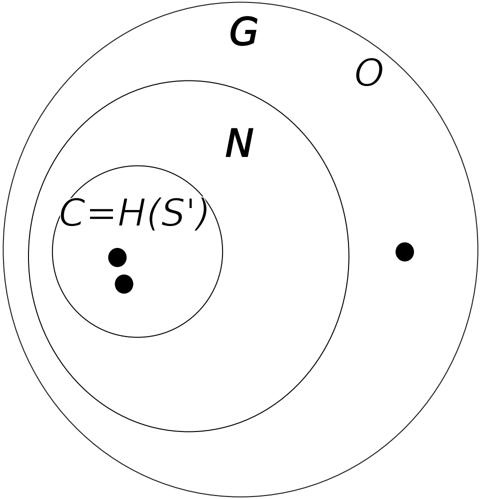

Algorithm

Algorithm 1: HULLSET(G)

Data: Diameter two graph $G = (V, E)$
Result: S' such that $N(H(S')) = V(G)$

$S' \leftarrow \emptyset$
$H \leftarrow \emptyset$
$O \leftarrow V$
while $O \neq \emptyset$ **do**
    $S' \leftarrow S' \cup \{v\} \mid v \in O$
    $H \leftarrow H(S')$
    $O \leftarrow V - N[H]$
end
return S'

Upper bound

Lemma

Consider the i-th iteration of Algorithm 1. If $O_i \neq \emptyset$*, then* $|H(S'_i)| \geq 2(|H(S'_{i-1})| + 1)$*.*

- $|H_i| \geq |H_{i-1}| + |N(u) \cap N_{i-1}| + 1$
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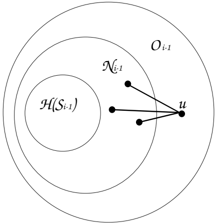

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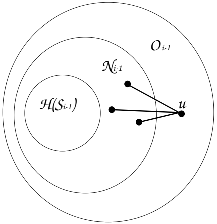

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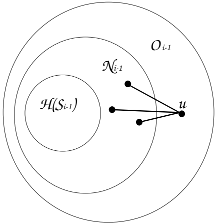

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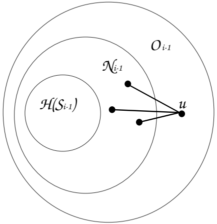

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Upper bound

Proposition

Algorithm 1 runs in at most $k = \lceil \log(\Delta(G) + 1) \rceil$ *iterations.*

Proof idea

- $|H(S_i')| \geq 2|H(S_{i-1}')| + 1$;
- Solving $T(n) \geq 2 \cdot T(n-1) + 1$;
- $|H(S_i')| \geq 2^i - 1$.
- O_i is empty when $|H(S_{i-1}')| > \Delta$;
- Then, $2^k \geq \Delta + 1$.
- Isolating k, we have that $k \geq \log(\Delta + 1)$.

Corollary

If G *is a biconnected diameter two graph, then* $h(G) \leq \lceil \log(\Delta(G) + 1) \rceil + 1$.

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Algorithm 1 runs in at most $k = \lceil \log(\Delta(G)+1) \rceil$ *iterations.*

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C_6-free diameter two graph

Definition

A graph is C_6-free if it contains no induced cycle with 6 vertices.

Theorem

If G is a biconnected C_6-free diameter two graph, then $h(G) \leq 4$.

Proof Idea

$S = \{a, b, c\}$ such that $| H(S) |$ is maximum.

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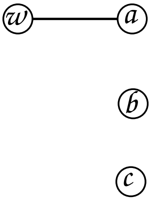

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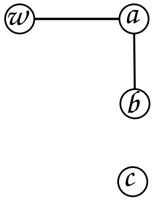

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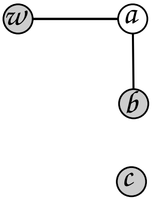

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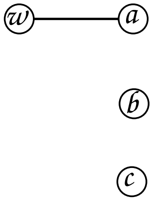

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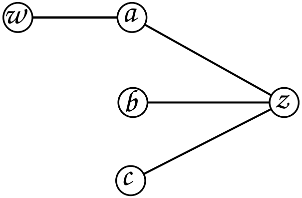

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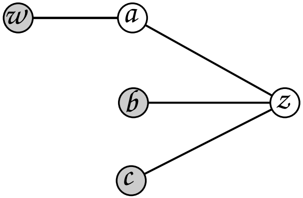

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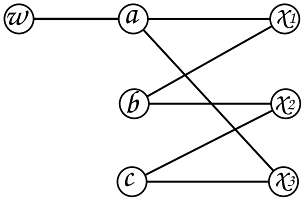

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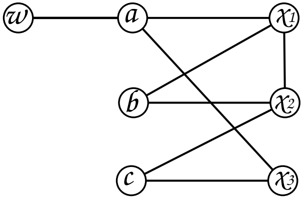

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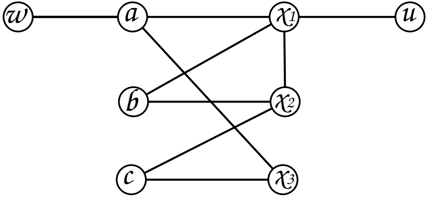

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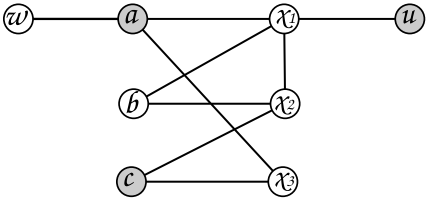

Strongly regular graphs

Definition

A graph $G(n, k, b, c)$, with n vertices, is *strongly regular* if is k-regular and every two adjacent vertices have b common neighbours and two non-adjacent vertices have c common neighbours.

Example

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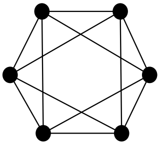
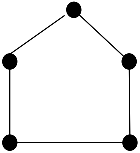

Octahedral (6,4,2,4)


C5 (5,2,0,1)

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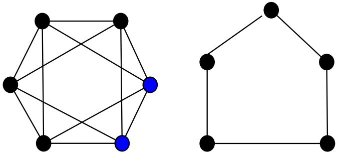

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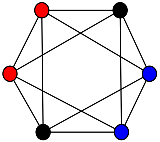

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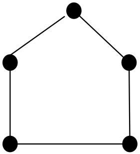

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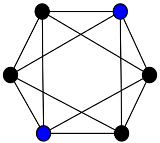

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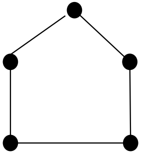


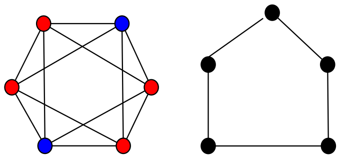
C5 (5,2,0,1)

Strongly regular graphs

Definition

A graph $G(n,k,b,c)$, with n vertices, is *strongly regular* if is k-regular and every two adjacent vertices have b common neighbours and two non-adjacent vertices have c common neighbours.

Example



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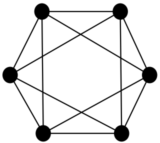
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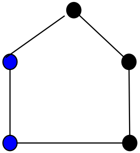
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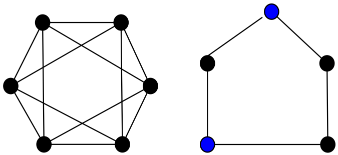

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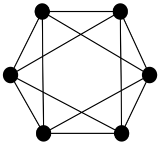

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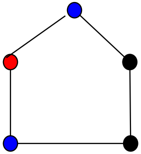

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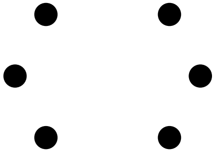
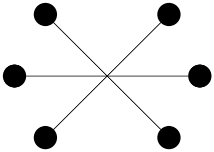
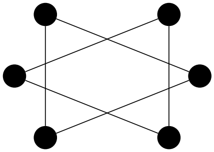


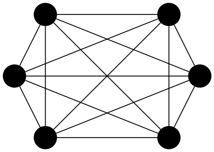
$G_1(6,0,0,0)$



$G_2(6,1,0,0)$



$G_3(6,2,1,0)$



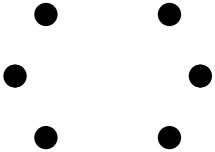
$G_4(6,5,4,0)$

Proposition

If G is a strongly regular graph $G(n,k,b,c)$ with $c=0$, then $h(G) \leq 2\omega(G)$.

Strongly regular graphs

Example



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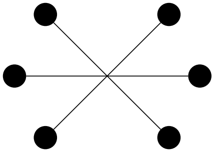

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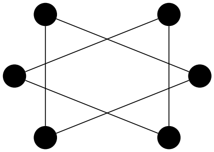

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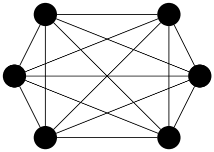

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Strongly regular graphs

Theorem

If G is a strongly regular graph $G(n, k, b, c)$ with $c > 0$, then

$$h(G) \leq \lceil \frac{k}{1+b} \rceil + 1,$$

Strongly regular graphs

Lemma

Consider the i-th iteration of Algorithm 1 when it is applied to a strongly regular graph $G(n,k,b,c)$ with $c>0$. If $O_i \neq \emptyset$, then $|H(S_i)| \geq (c+1) \times (|H(S_{i-1})|+1)$.

Proposition

If Algorithm 1 is applied to a strongly regular graph $G\ (n,k,b,c)$, it runs in at most $i = \lceil \log_{c+1}(kc+1) \rceil$ iterations.

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Comparative table

Graph G	$h(G)$	Corol. 3	Theo. 2	Theo. 3
C_5 (5,2,0,1)	3	3	3	3
Petersen (10,3,0,1)	3	3	4	3
Clebsch (16,5,0,2)	3	4	6	3
(4,4)-Hook (16,6,2,2)	2	4	3	3
Hoffman-Singleton (50,7,0,1)	4	4	8	4
Gewirtz (56,10,0,2)	3	4	11	4
Brouwer-Haemers (81,20,1,6)	2	5	11	3
M_{22} (77,16,0,4)	2	5	17	4
$H(2,10)$ (100,18,8,2)	2	5	3	4
Higman-Sims (100,22,0,6)	2	6	23	4
Cameron (231,30,9,3)	2	6	4	4
Berlekamp-van L. (243,22,1,2)	3	6	12	4
McLaughlin (275,112,30,56)	2	8	5	3
Grassmann $J_2(6,2)$ (651,90,33,9)	2	8	4	4
G2(4) (416,100,36,20)	2	8	4	3

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Thank you for your attention!