Dual Parametrization of Weighted Coloring

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joint work with
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Weighted Coloring

We are given a graph G together with a weight function $w : V(G) \to \mathbb{R}^+$.

A (proper) k-coloring of G is a partition $c = (S_i)_{i \in [1,k]}$ of $V(G)$ into k stable sets $S_1, \ldots, S_k$.

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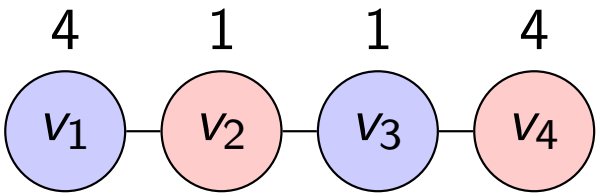
For a positive integer r, we define

$$\sigma(G, w; r) \;=\; \min\{w(c) \mid c \text{ is a proper } r\text{-coloring of } G\}.$$

Some authors call it Max Coloring!

Example



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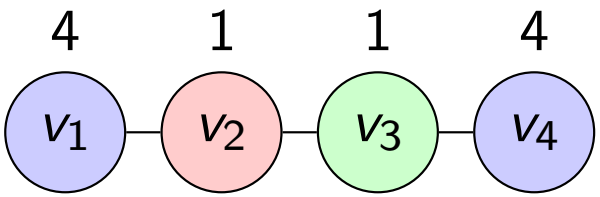

$$w(c) = w(S_1) + w(S_2) + w(S_3) = 6$$

$$\sigma(P_4, w) = 6$$

$$\sigma(P_4, w; 2) = 8$$

Remark

If $G = (V, E, w)$ has all the weights equal to 1:

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- $\sigma(G, w) = \chi(G)$;
- Hardness results hold for $\sigma(G, w)$.

Some complexity results up to 2006

Polynomial

- Bipartite graphs with 2 different weights [Demange et al. 2002]
- P_5-free bipartite graphs [Demange et al. 2005]
- P_4-**free graphs** (i.e., Cographs) [Demange et al. 2002]

NP-complete

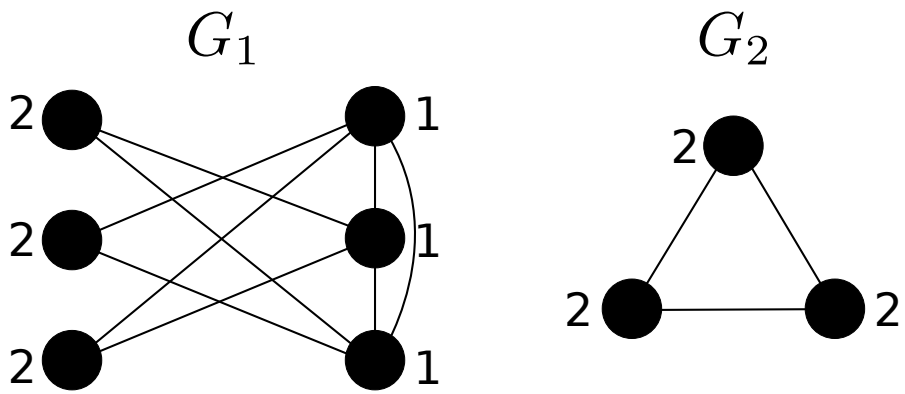
- Interval graphs [Escoffier et al. 2005]
- Bipartite graphs with $\Delta(G) \leq 14$ [Demange et al. 2002]
- Triangle-free planar graphs with $\Delta(G) \leq 4$ [Demange et al. 2005]
- Split graphs [Demange et al. 2002]
- $\Rightarrow$ P_5-**free graphs**

Approximation algorithms, etc.

P_4-sparse graph

$G = (V, E)$ is P_4-sparse if $\forall H \subseteq V$ s. t. $|H| \leq 5$ induces at most one P_4.

Problem with the disjoint union! $\rightarrow$ an optimal coloring of $G = G_1 \cup G_2$ is not given by optimal colorings of G_1 and G_2.



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Theorem (A., Linhares Sales, Sau, 2007)

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There is a 2-approx. algorithm to $\sigma(G, w)$, for P_4-sparse graphs G.

Hajós-like Theorem

Theorem (A., Linhares Sales, 2007)

$\sigma(G, w) \geq k$ if, and only if, G contains a weighted k-constructible subgraph.

Open problem | Is Weighted Coloring **polynomial** on trees/forests?
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Some partial results:

- PTAS on bounded treewidth graphs. [Escoffier, Monnot, Paschos. 2006]
- Polynomial on the class of trees where vertices with degree at least three induce a stable set. [Kavitha, Mestre. 2012]
  - Some improvements for subclasses of trees in [Benkoczi, Dahal, Gaur, 2016]

Complexity of weighted coloring on trees (or forests)

On an n-vertex graph of treewidth t, $\sigma(G, w; r)$ can be computed in time

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$\Longrightarrow$ Weighted Coloring can be solved on forests in time $n^{O(\log n)} = 2^{O(\log^2 n)}$ (quasi-polynomial).

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That is, the running time $n^{O(\log n)}$ is tight under the ETH.

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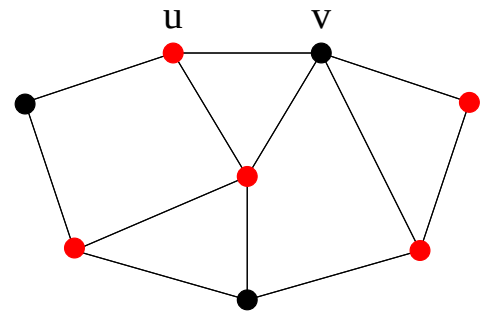
The theory of parameterized complexity is built based on FPT $\neq$ W[1].

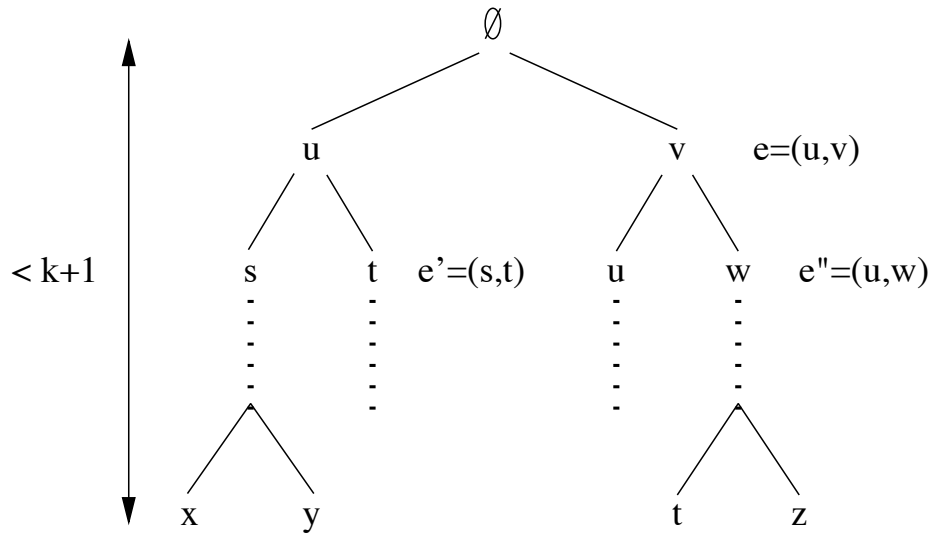
W[1]-hardness: strong evidence of not being FPT.
W[2]-hardness: even more!

Example of an FPT Algorithm

k-Vertex Cover: Does a graph G contain a set $S \subseteq V(G)$, with $|S| \leq k$, covering $E(G)$?



Easy branching rule: Let (G, k) be an instance and let $e = \{u, v\} \in E(G)$. Branch into the two smaller instances $(G - u, k - 1)$ and $(G - v, k - 1)$.

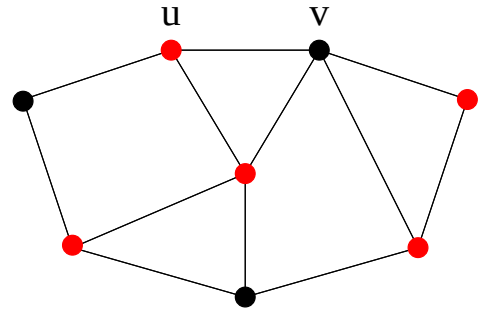


Running time: $O(2^k \cdot (m + n))$

Here, $n = |V(G)|$ and $m = |E(G)|$

Example of Kernelization Algorithm

k-Vertex Cover: Does a graph G contain a set $S \subseteq V(G)$, with $|S| \leq k$, covering $E(G)$?



Reduction rule: over instance (G, k).

- If $\exists v$ s.t. $d(v) > k$, then (G, k) is YES iff $(G - v, k - 1)$ is YES.

$\Rightarrow$ we build an equivalent instance (G', k') with $\Delta(G') \leq k$!

If (G', k') has more than k^2 edges, the answer is NO.

Otherwise, G' has at most $2k^2$ vertices and at most k^2 edges.

G' is a kernel!

$\Rightarrow$ One can then make brute force on (G', k').

Running time: $O(2^{2k^2} \cdot (m + n))$

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Corollary (A., Baste, Sau)

Assuming ETH, there is no algorithm that, given a weighted tree (G, w) and a positive integer r, computes $\sigma(G, w; r)$ in time $f(r) \cdot n^{o(r)}$ for any computable function f.

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The dual parameterization has proved to be useful for Vertex Coloring, Grundy Coloring, and b-Coloring.

[Chor, Fellows, Juedes. 2004]

[Havet, Sampaio. 2013]

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- Algorithm running in time $2^{\mathcal{O}(k \log k)} \cdot n^{\mathcal{O}(1)}$.
- Asked "what size of kernel can be achieved for the problem?"

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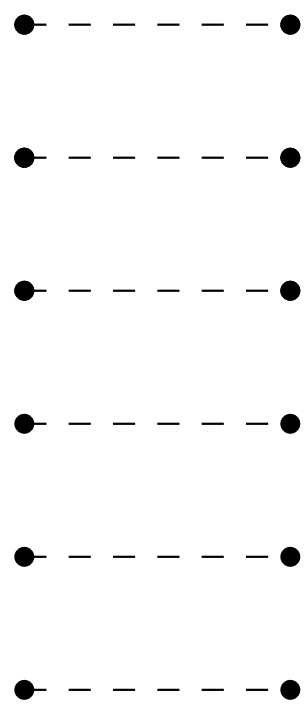
   - Kernel on Subclasses of split graphs, with lower bounds on the degrees.

An exponential kernel for Dual Weighted Coloring

We start with an idea already used, in particular, by [Escoffier. 2016]

An exponential kernel for Dual Weighted Coloring

maximum unweighted antimatching $\overline{M}$



We compute (in poly time) a maximum unweighted antimatching $\overline{M}$.

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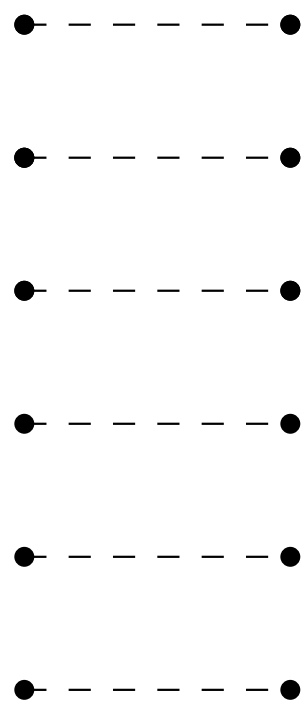

We may assume $|\overline{M}| \leq k-1$

If $|\overline{M}| \geq k$, we are dealing with a Yes-instance.

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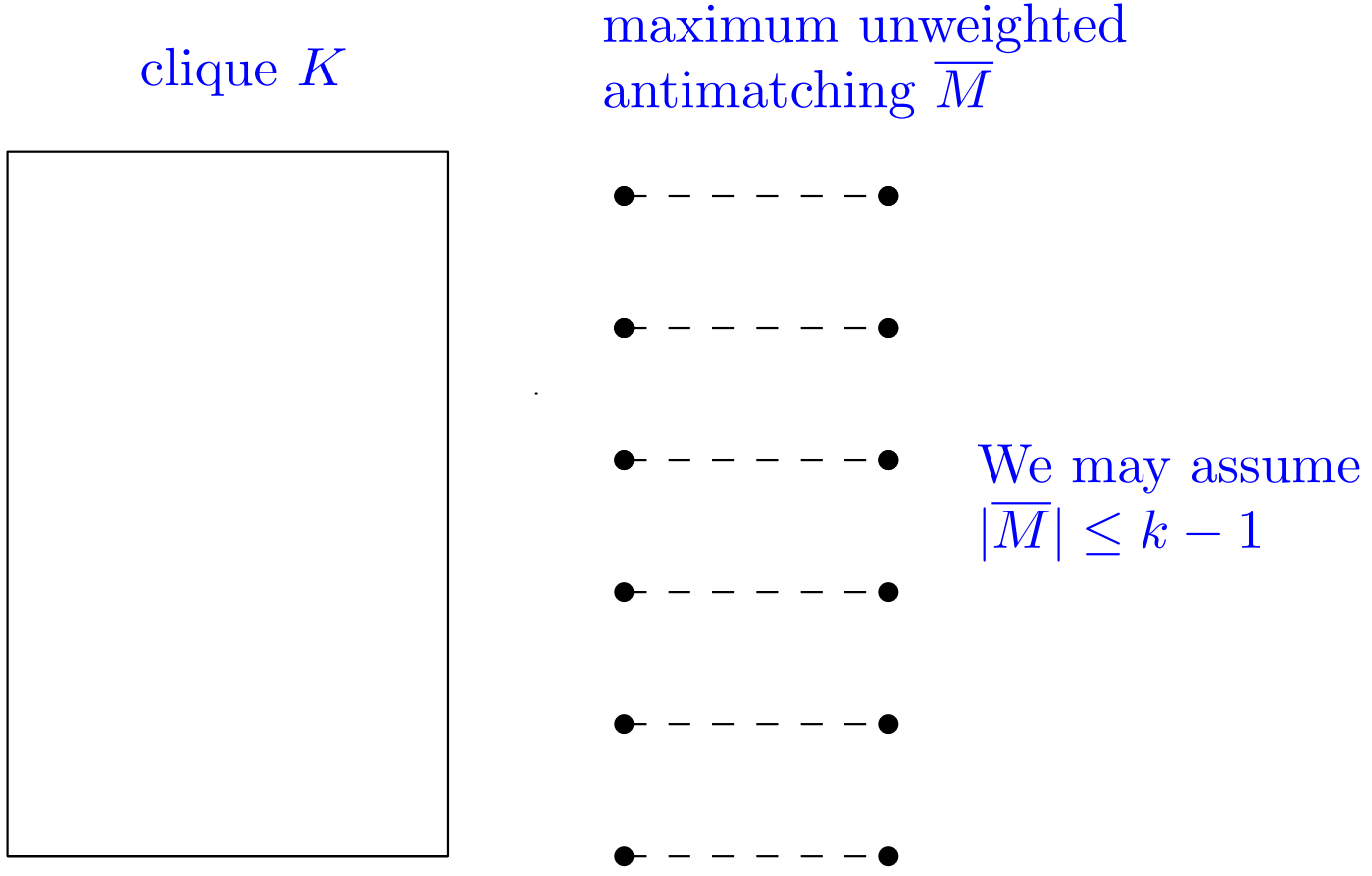

The complement of the antimatching $\overline{M}$ induces a clique K.

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clique K

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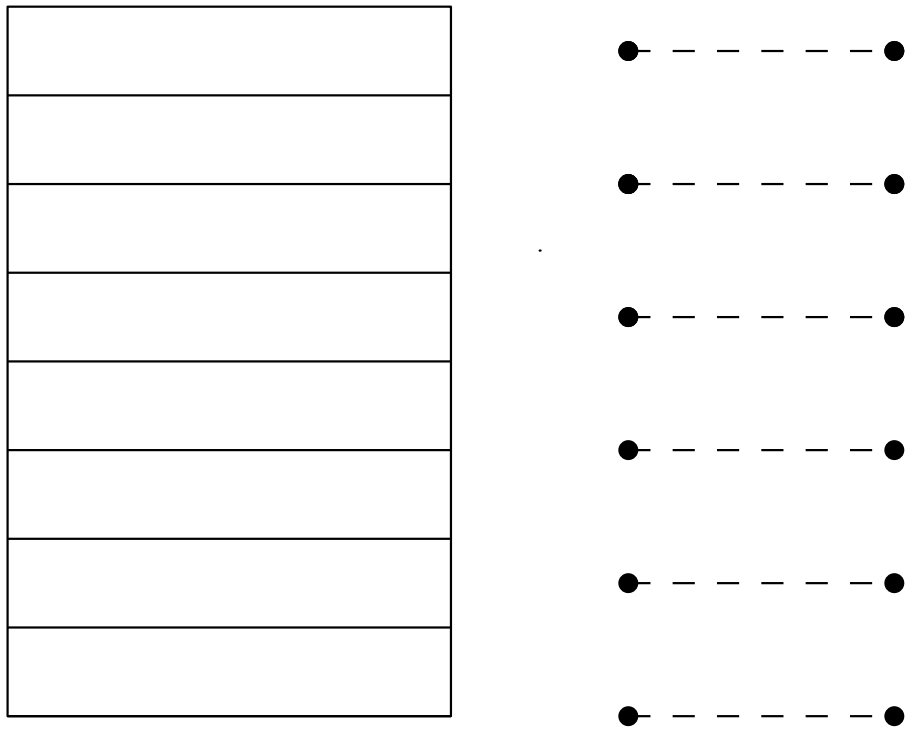
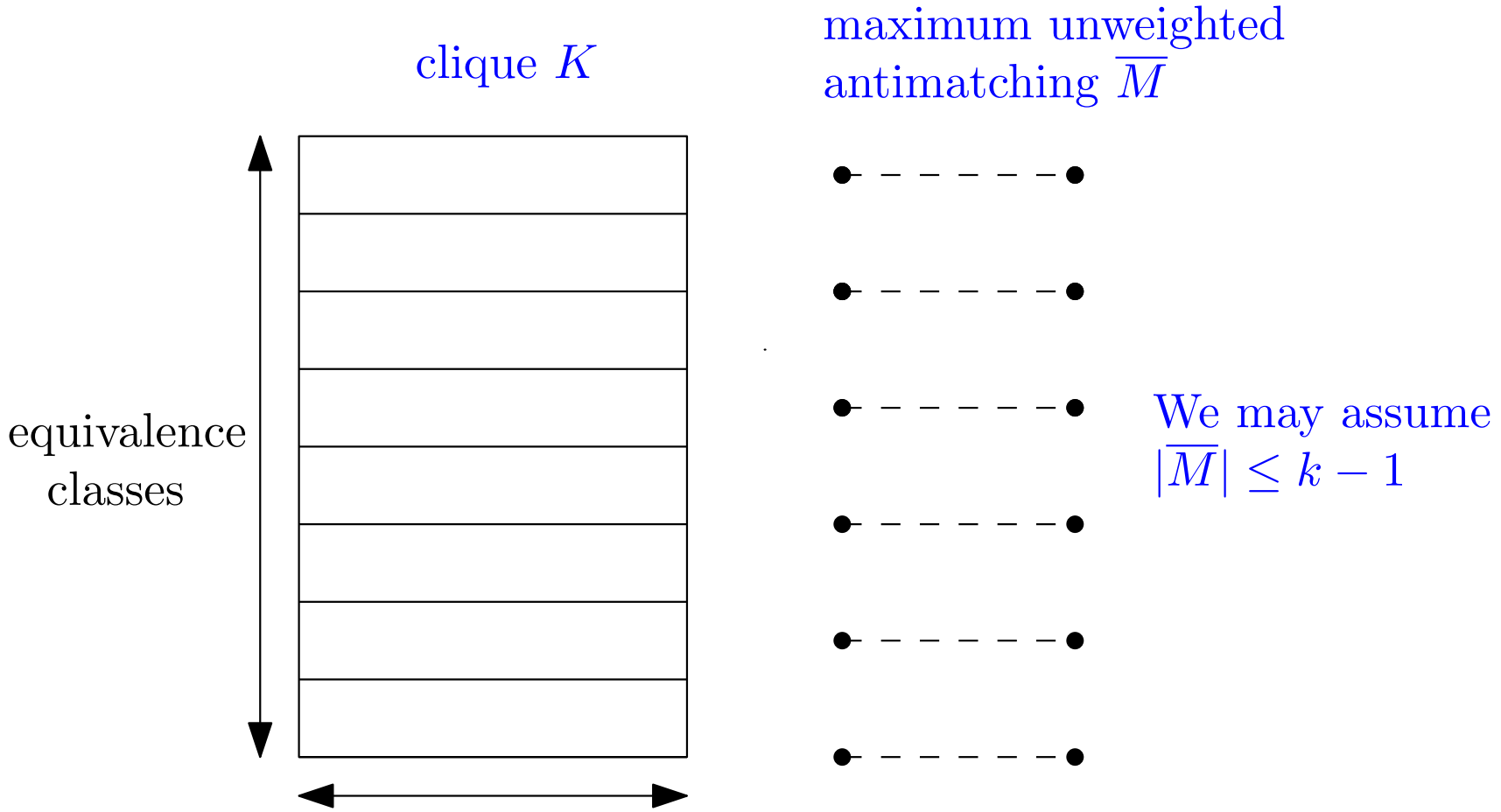


equivalence classes

We may assume $|\overline{M}| \leq k-1$

We partition K into equivalence classes w.r.t. the neighborhood in $\overline{M}$.

An exponential kernel for Dual Weighted Coloring



We have to bound the number and the size of the equivalence classes.

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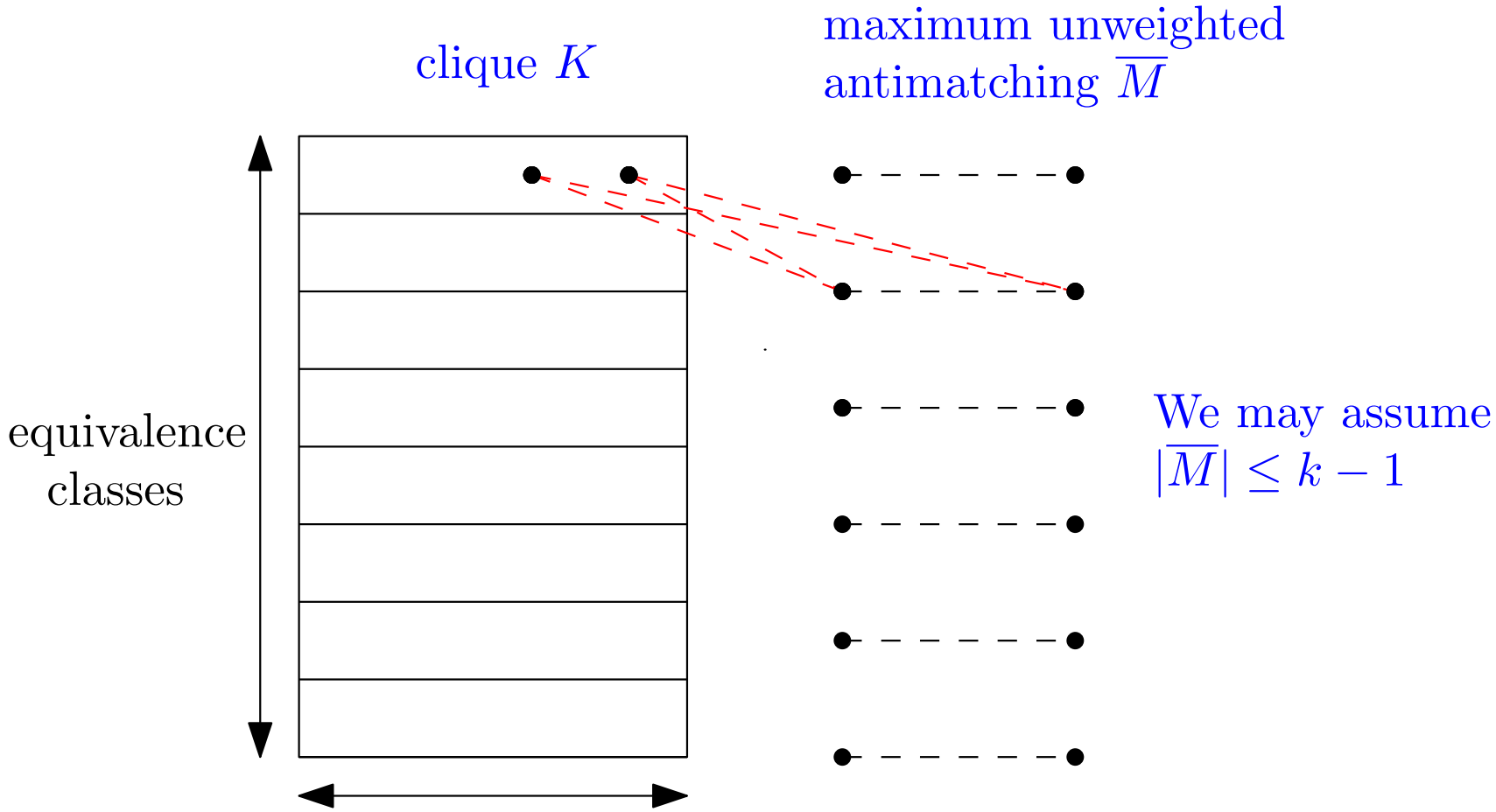

An equivalence class cannot miss both endvertices of a non-edge.

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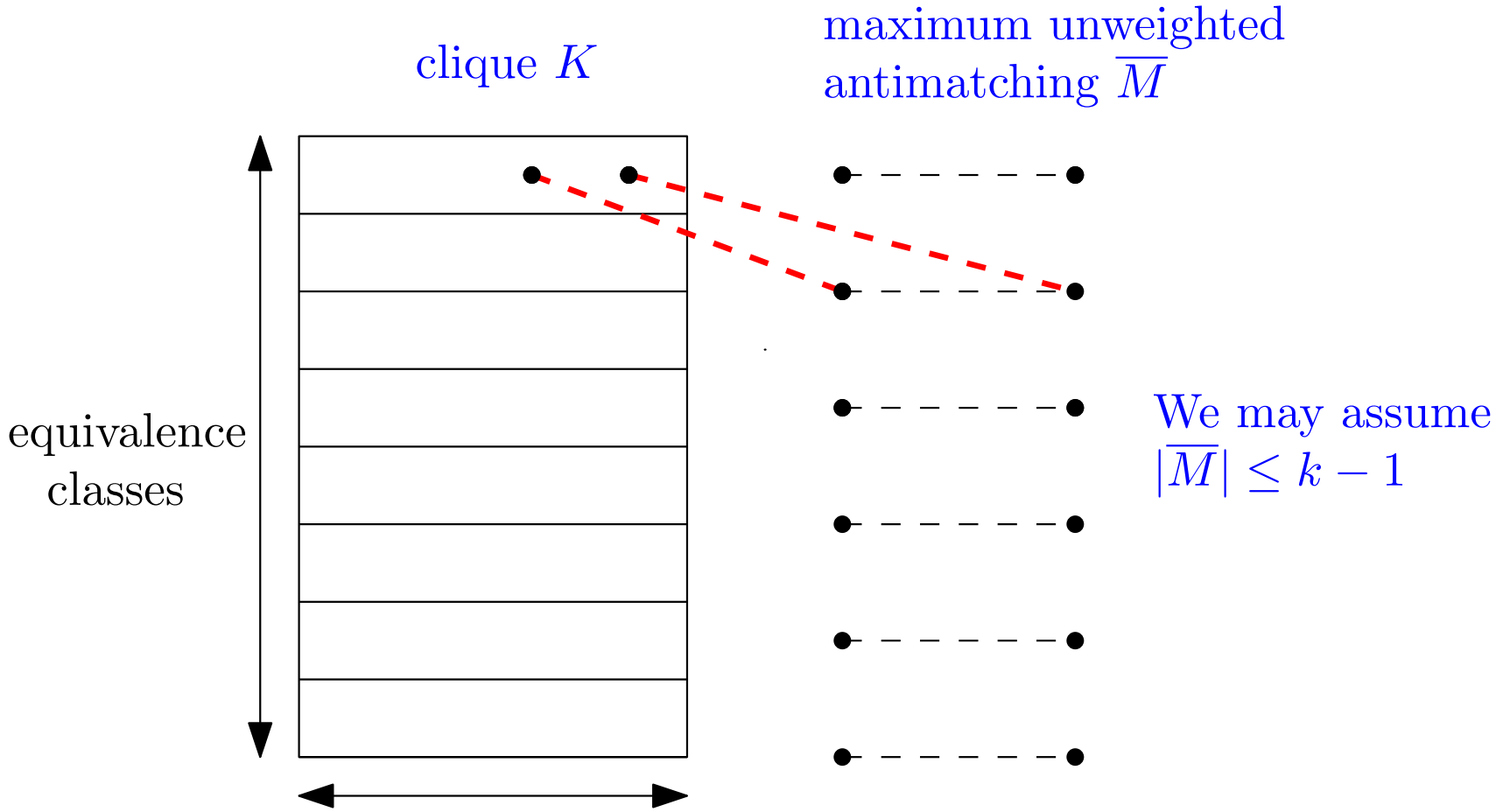

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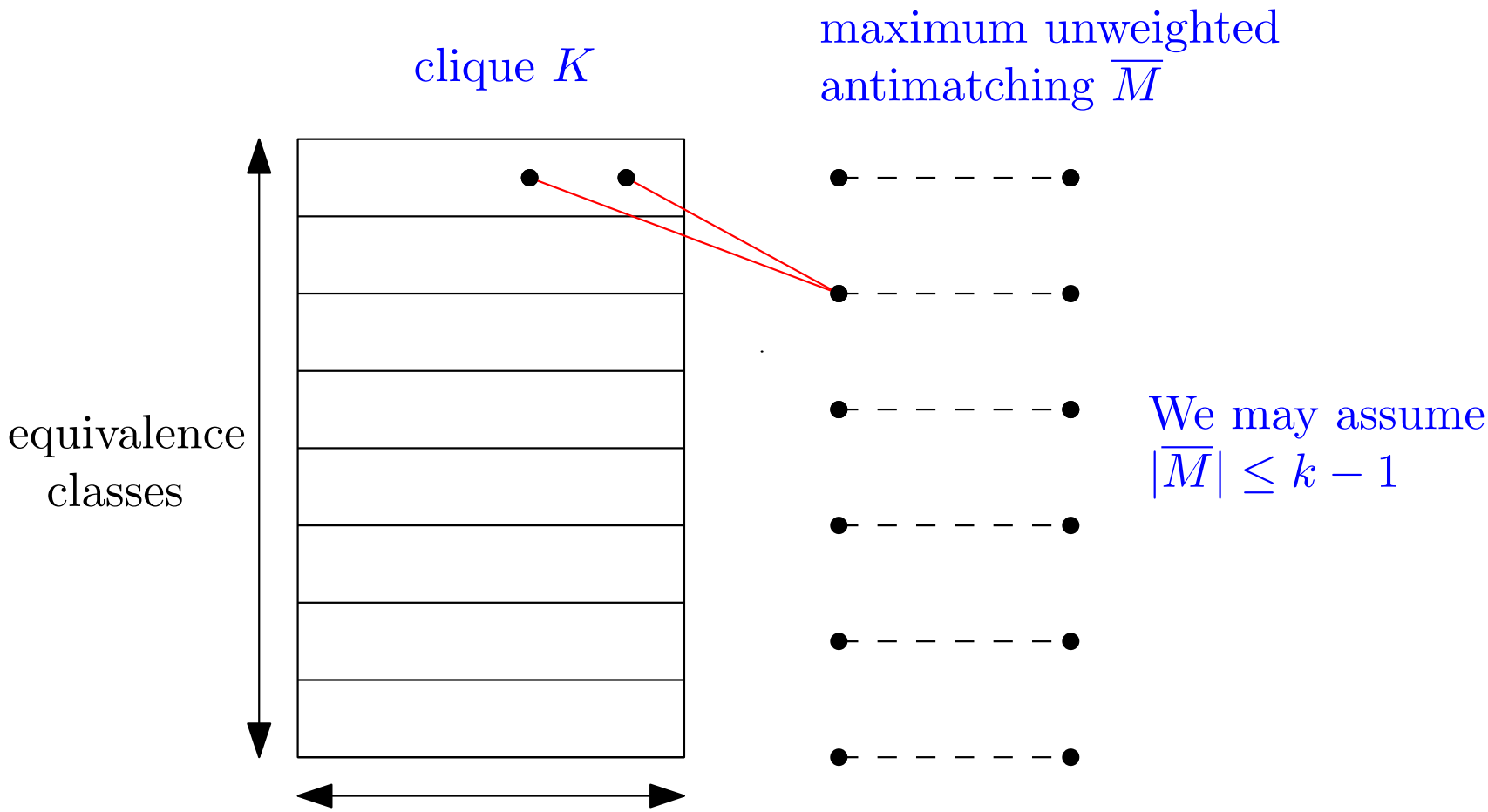

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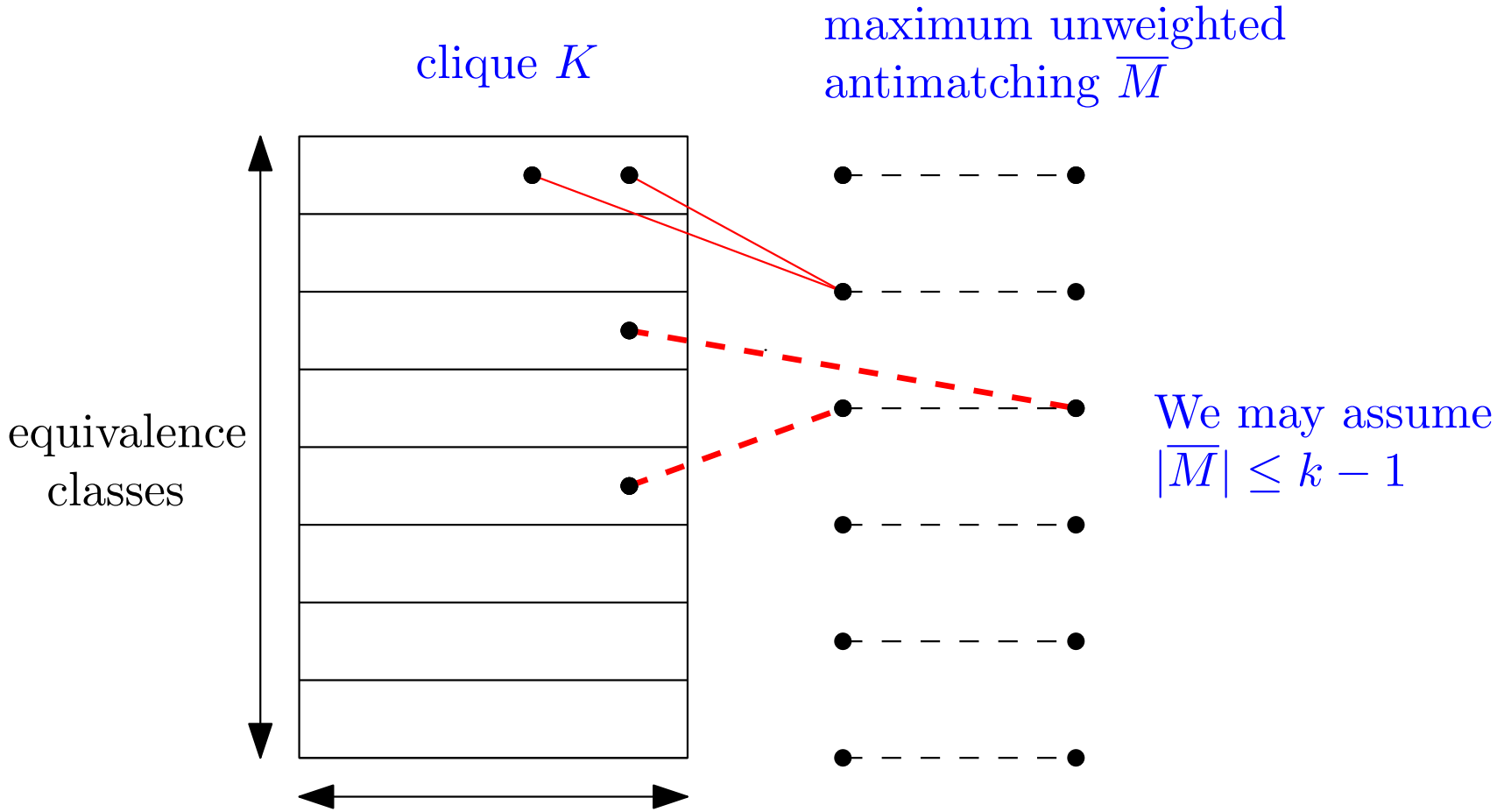

Two equivalence classes cannot miss different endvertices of a non-edge.

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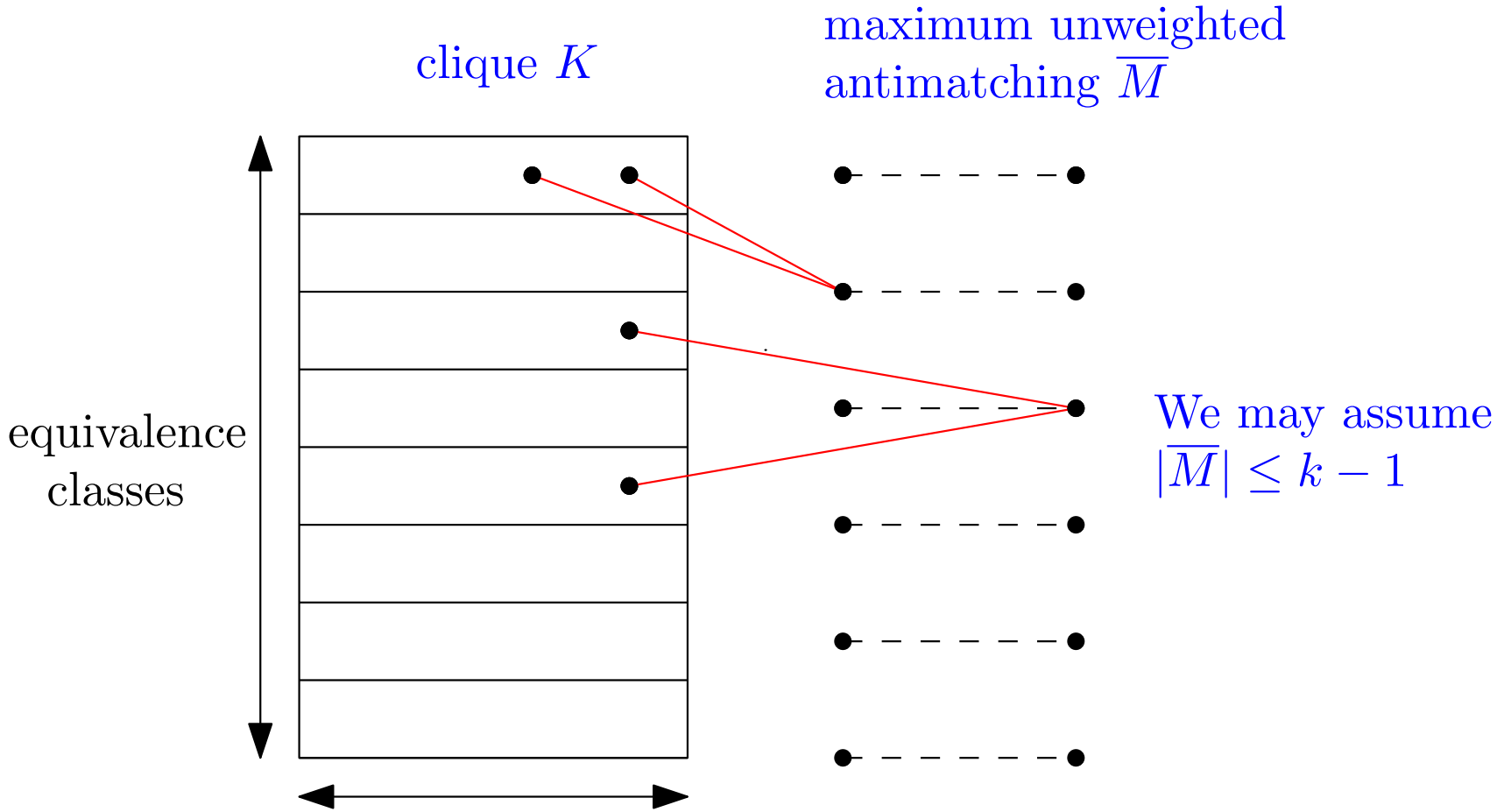

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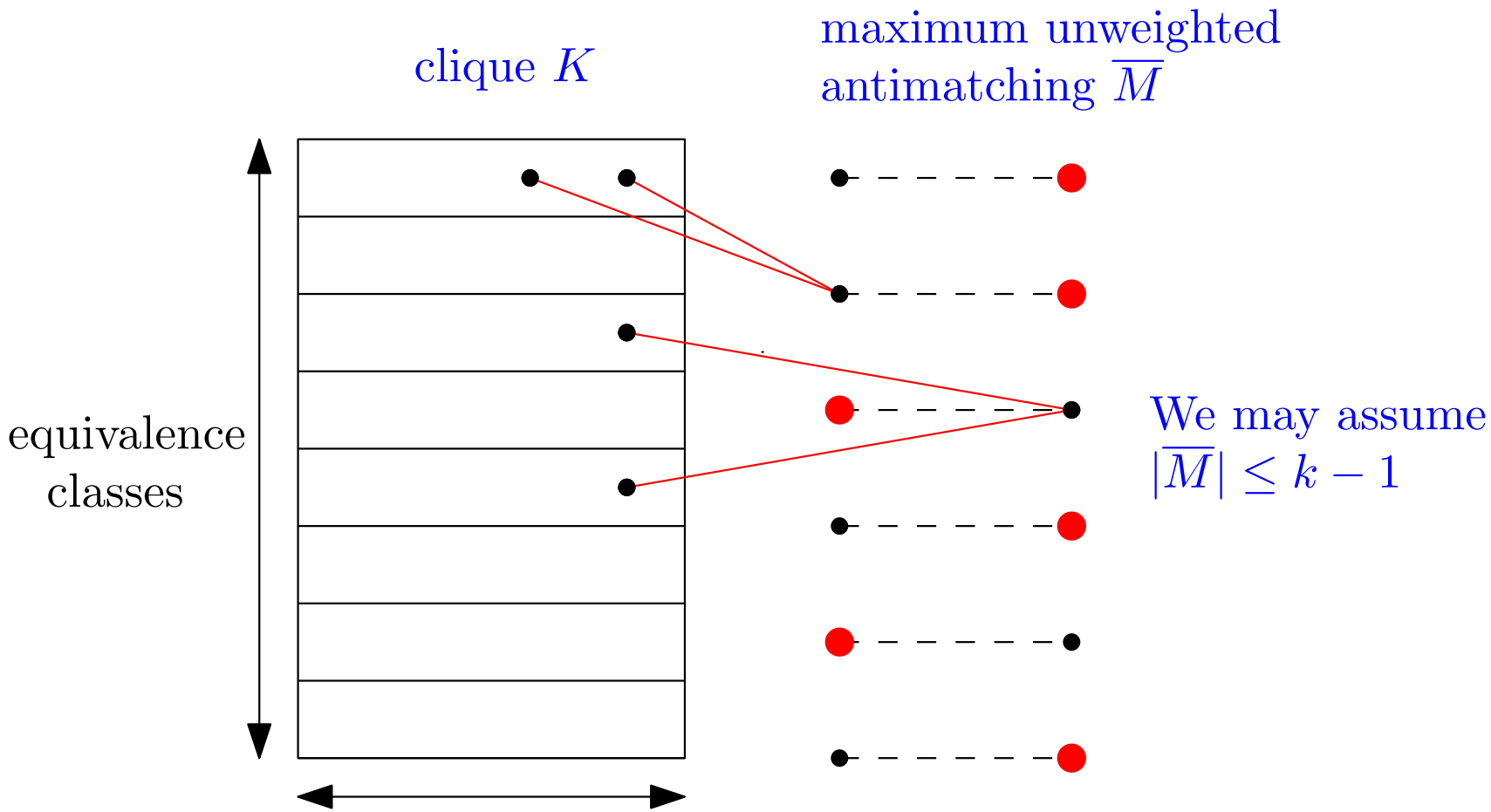

Every non-edge has a unique "potential non-neighbor" for the classes.

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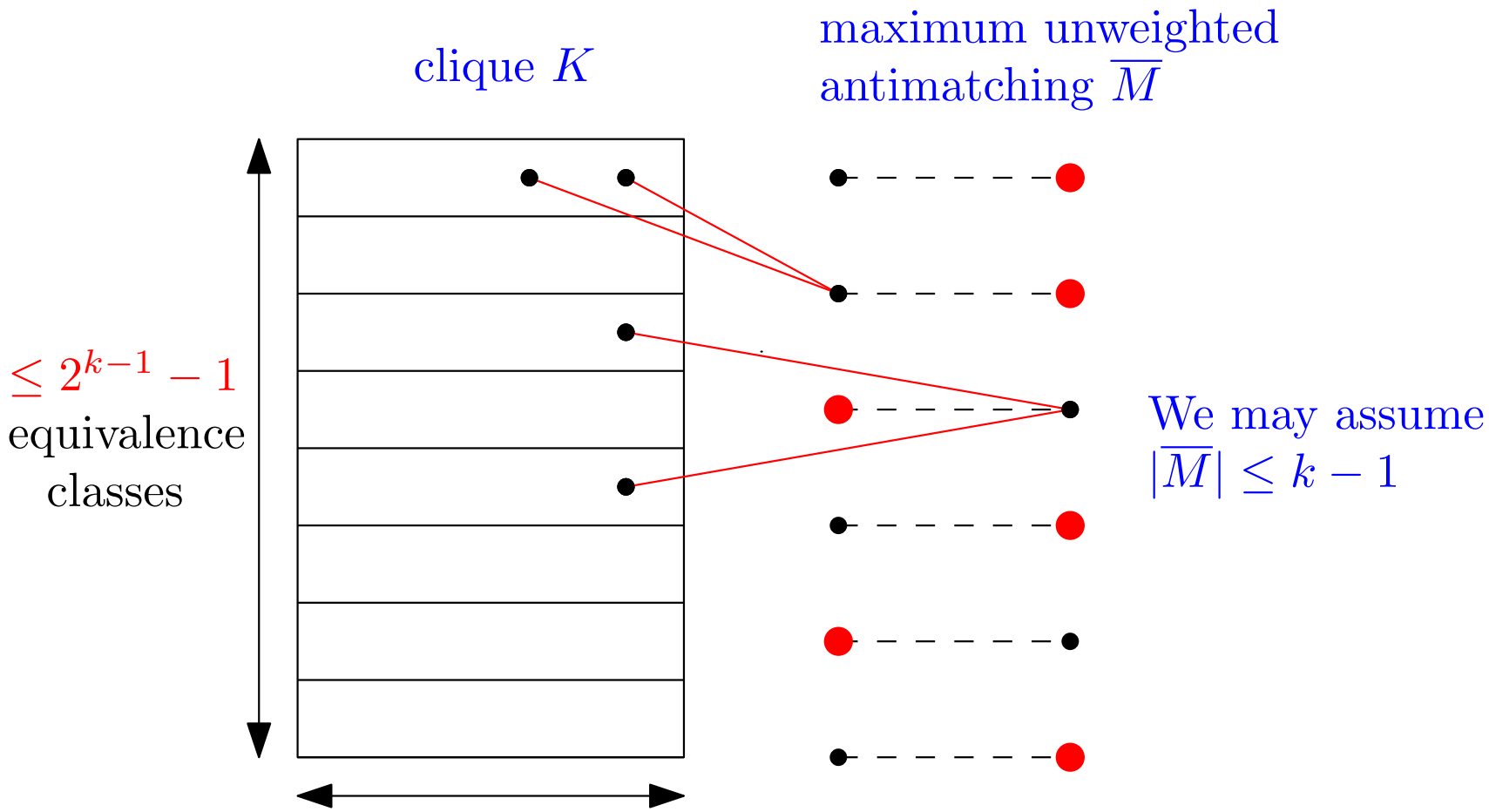

Hence, the number of equivalence classes is at most $2^{k-1} - 1$.

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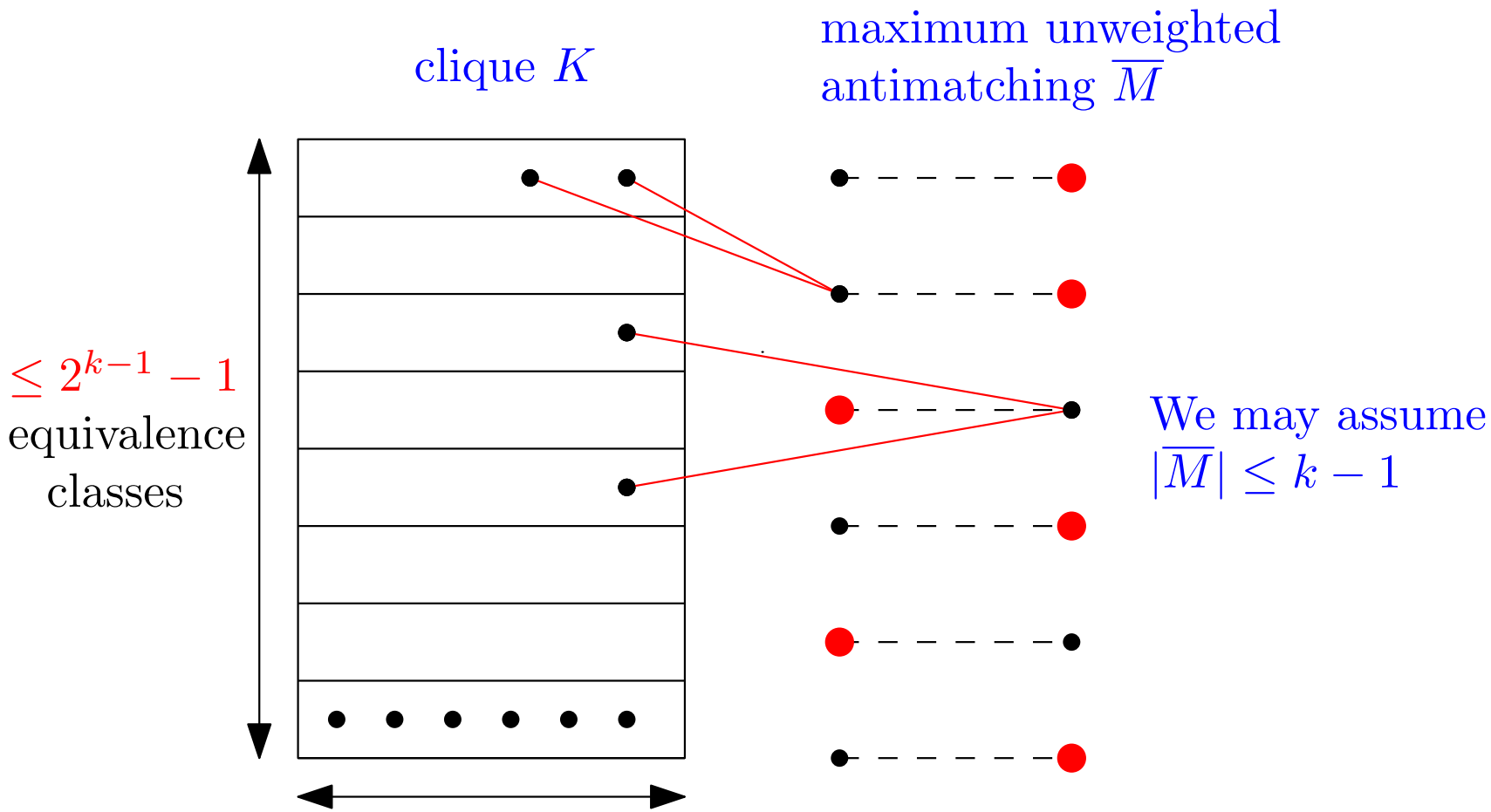

It remains to bound the size of each equivalence class.

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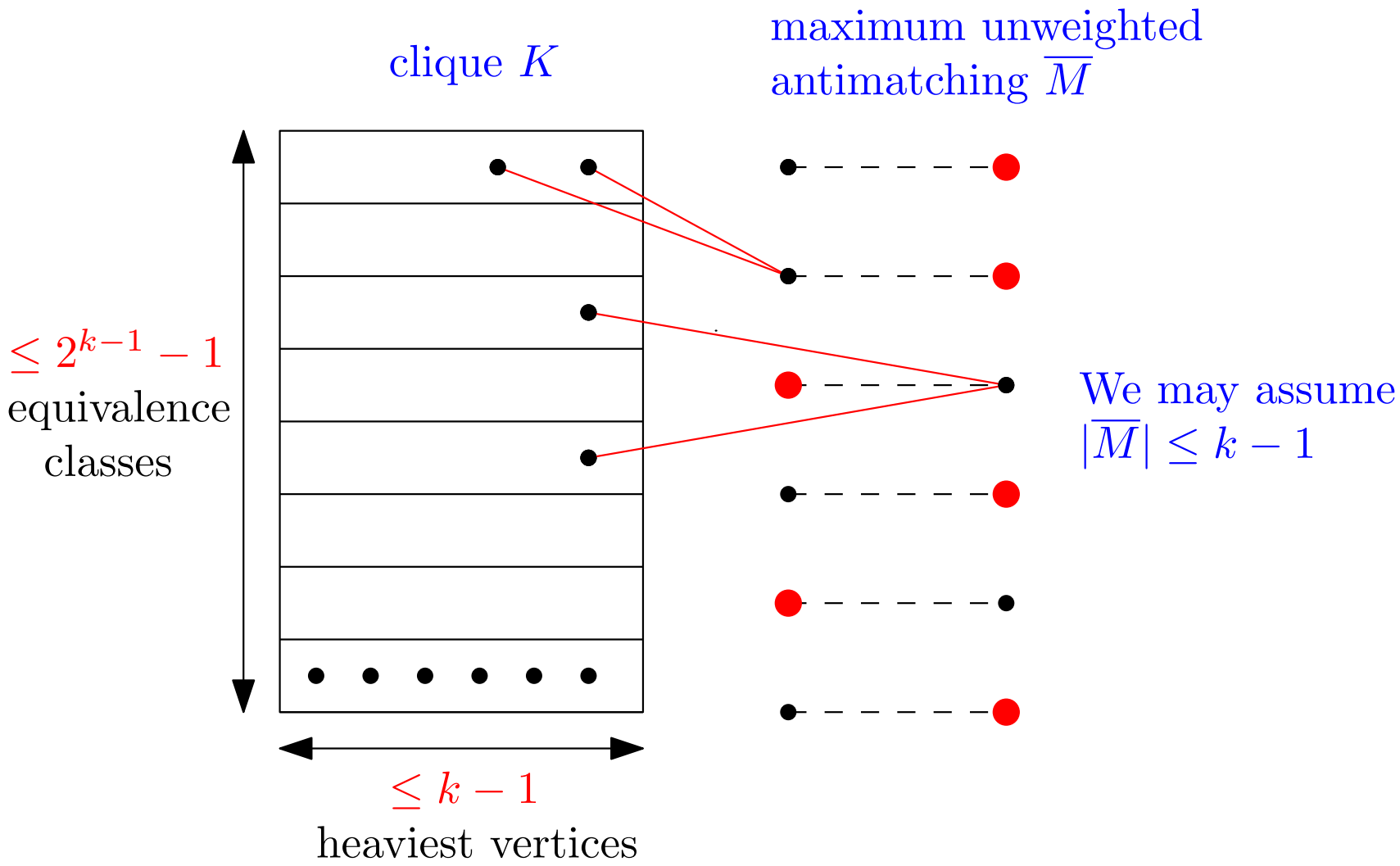

It is safe to keep only the $k - 1$ heaviest vertices of each equivalence class.

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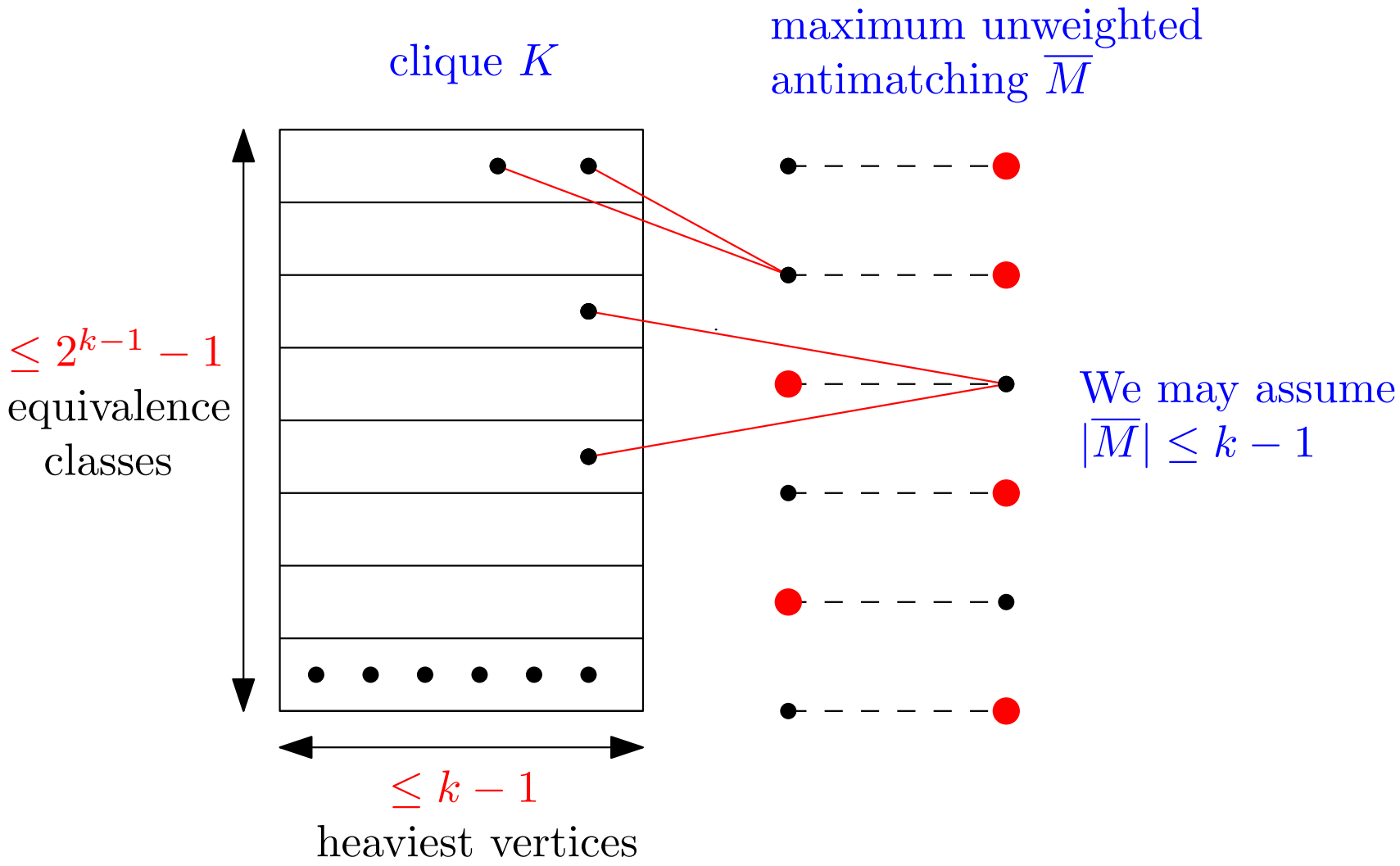

Thus, $|V(G)| \leq |V(\overline{M})| + |K| \ \leq \ 2(k-1) + (2^{k-1} - 1)(k-1)$.

Ruling out polynomial kernels

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Let $A, B \subseteq \Sigma^* \times \mathbb{N}$ be two parameterized problems.

A polynomial parameter transformation from A to B is an algorithm s.t.:

1. (x, k) is a Yes-instance of $A \Leftrightarrow (x', k')$ is a Yes-instance of B.
2. $k' \leq \text{poly}(k)$.

[Bodlaender, Thomassé, Yeo. 2011]

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Our reduction is an appropriate modification of a reduction to prove the NP-hardness of Weighted Coloring on split graphs:

only the weights change.

[Demange, de Werra, Monnot, Paschos. 2002]

Polynomial parameter transformation from Set Cover

We are given an instance $(U, \mathcal{S}, k, \ell)$ of Set Cover,
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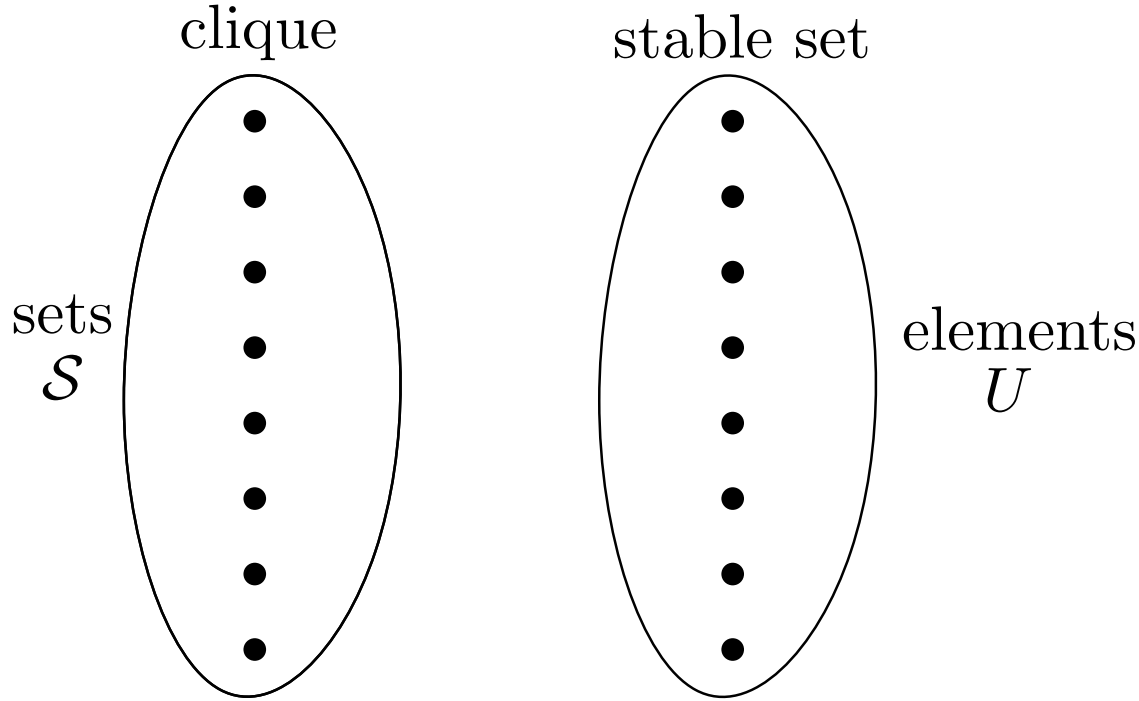

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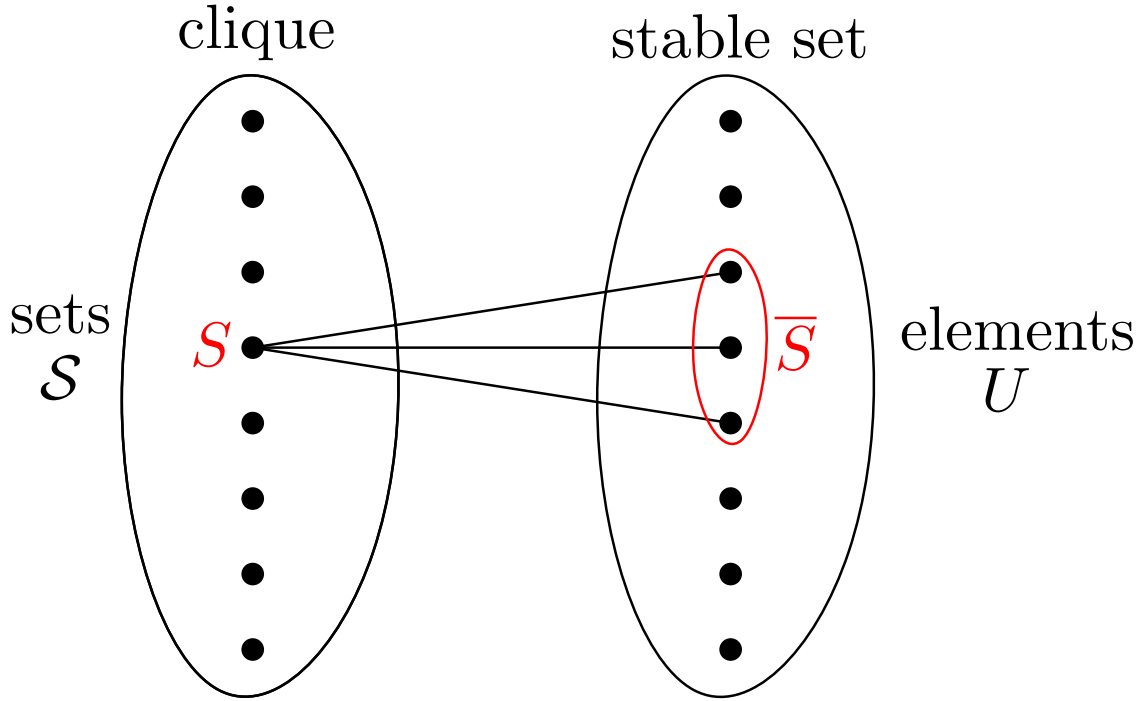

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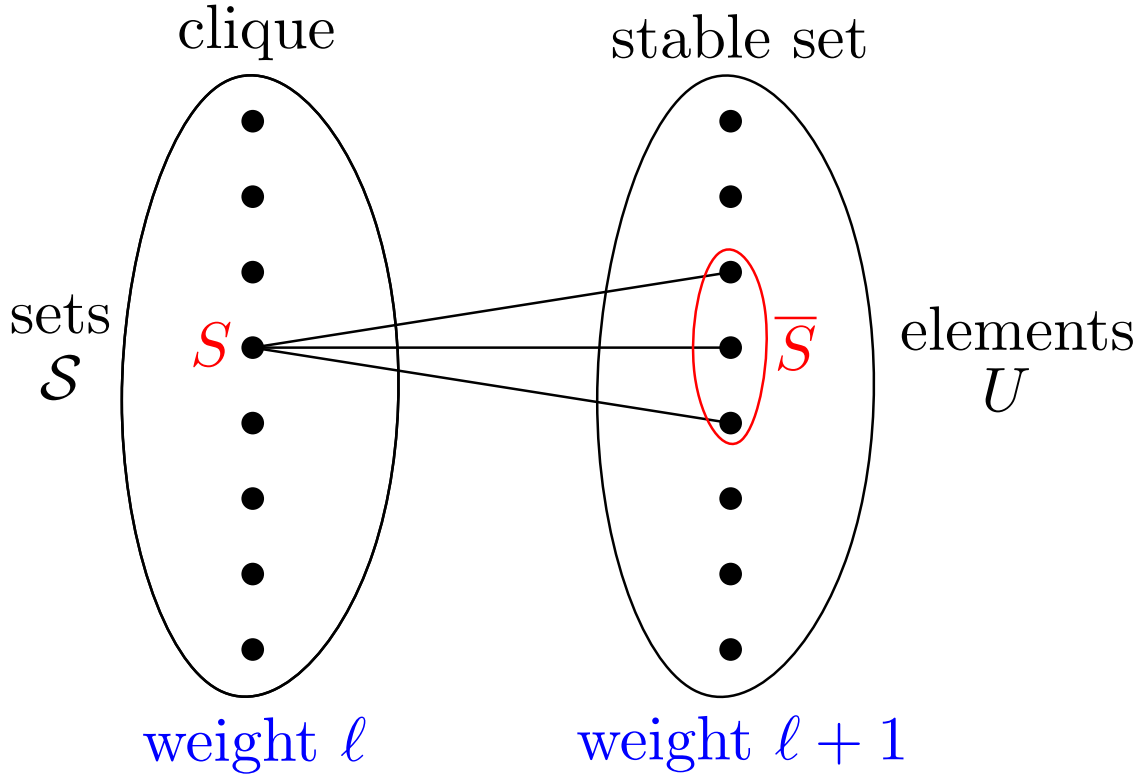

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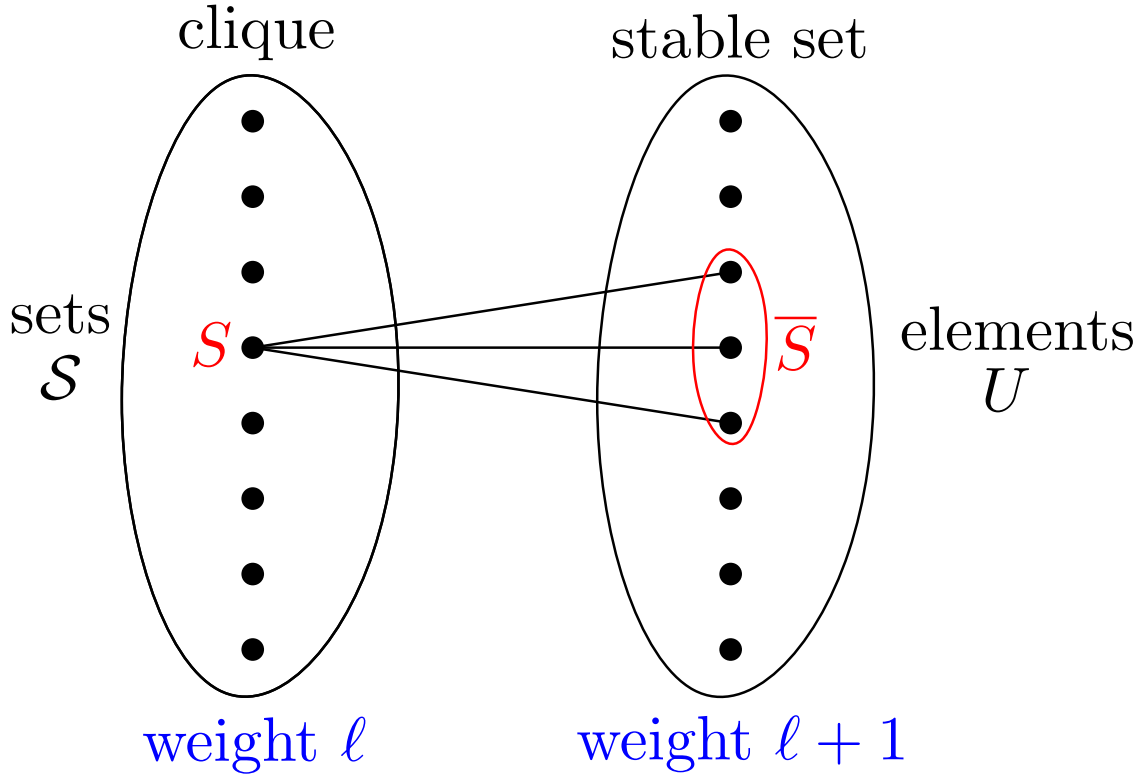

We build and instance (G, w, k'), with $k' = k(\ell + 1) - \ell = \mathcal{O}(k^2)$.

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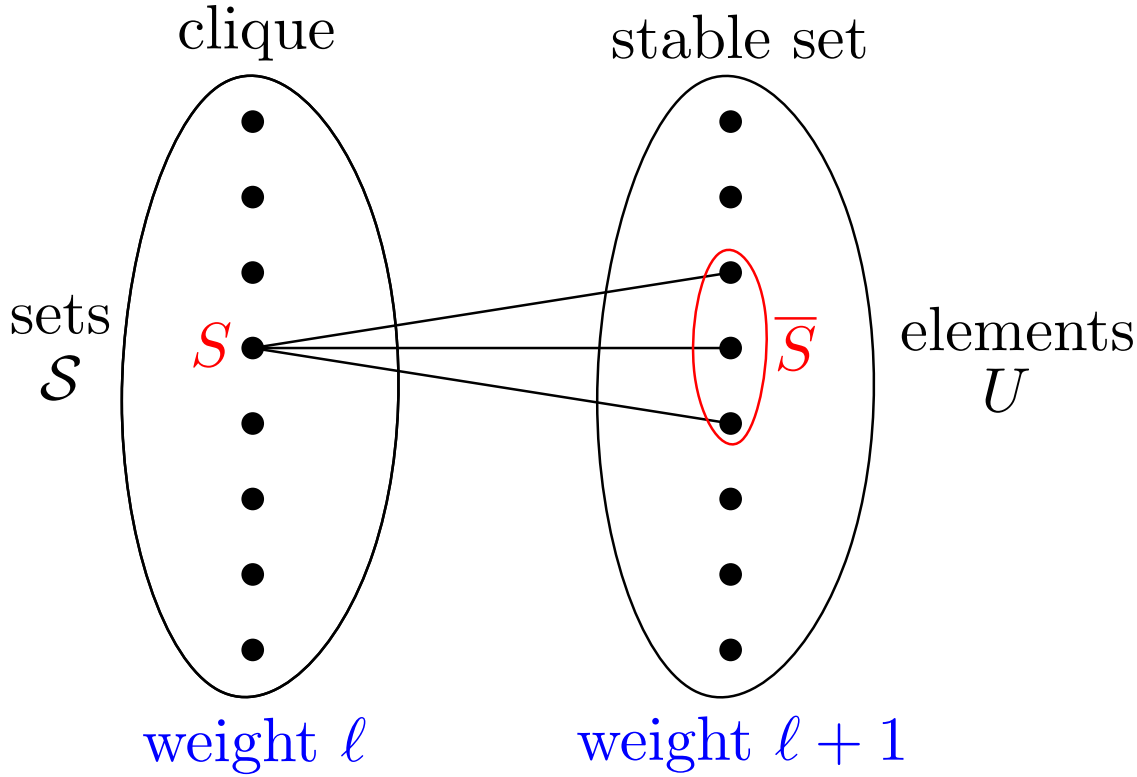

$(U, \mathcal{S}, k, \ell)$ satisfiable $\Leftrightarrow$ $\sigma(G, w) \leq \sum_{v \in V(G)} w(v) - k' = |\mathcal{S}| \cdot \ell + \ell$.

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Furthermore, for any $\varepsilon > 0$, a kernel with $\mathcal{O}(k^{\frac{d-3}{2}-\varepsilon})$ vertices does not exist unless NP $\subseteq$ coNP/poly.

Using the kernel lower bound for d-Set Cover. [Hermelin, Wu. 2012]

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- Can the cubic kernel on interval graphs be improved?
- Identify other classes of (dense) graphs allowing for polynomial kernels.

Thank you!