The (Parameterized) complexity of Equitable Coloring

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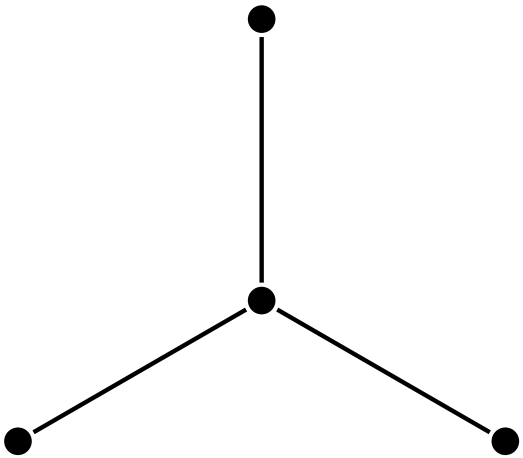

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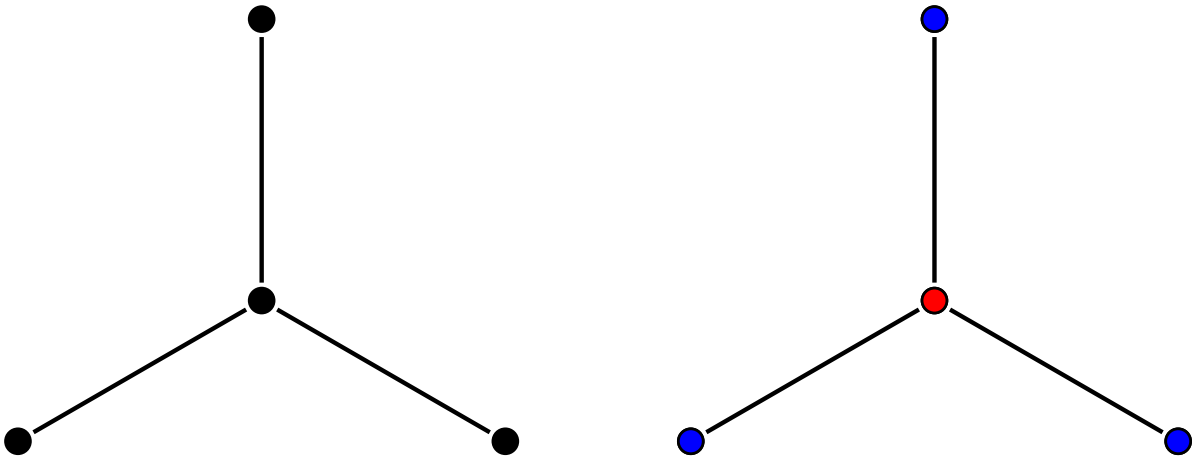

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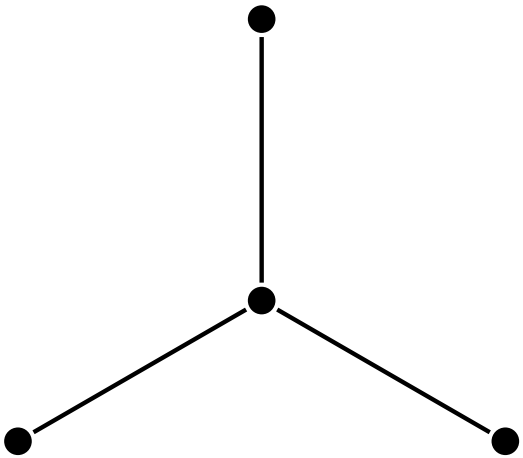

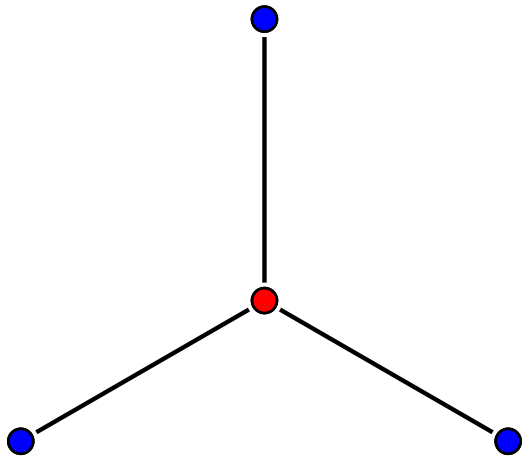

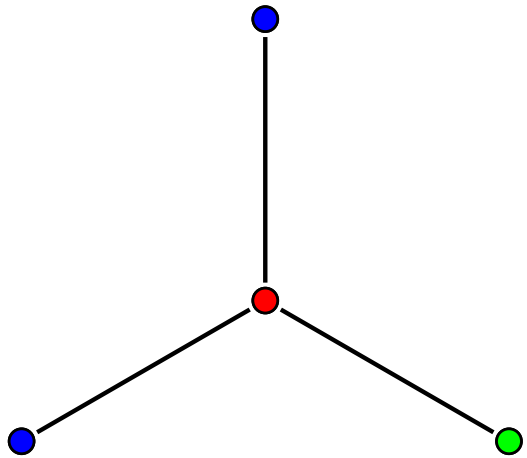

Applications

- Municipal garbage collection;
- Processor task scheduling;
- Communication control;
- Server load balancing.

Some important stuff I

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Equitable Coloring Conjecture

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Any graph G is equitably k-colorable if $k \geq \Delta(G) + 1$. Equivalently, $\chi^*_=(G) \leq \Delta(G) + 1$.

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For every connected graph G which is not a complete graph, an odd-hole nor $K_{2n+1,2n+1}$, for any $n \geq 1$, $\chi^*_=(G) \leq \Delta(G)$ holds.

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Ko-Wei Lih. "Equitable coloring of graphs". In: *Handbook of combinatorial optimization*. Springer, 2013, pp. 1199–1248

The story so far...

Class	Complexity
Trees	P
Forests	P
Bipartite	NP-complete, even if $k = 3$
Co-bipartite	P
Cographs	NP-complete, P for each fixed k
Bounded Treewidth	P
Chordal	NP-complete
Block	?
Split	P
Unipolar	?
Interval	NP-complete
Co-interval	P

After this talk

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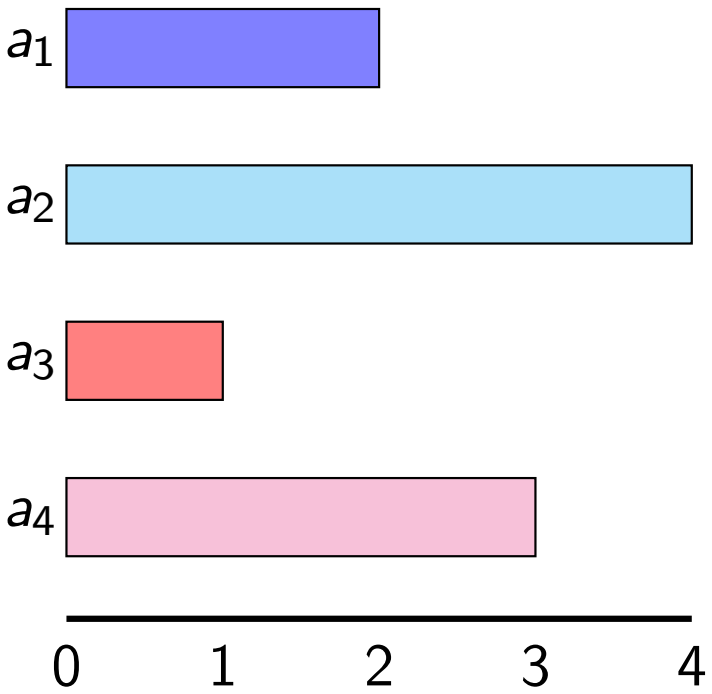

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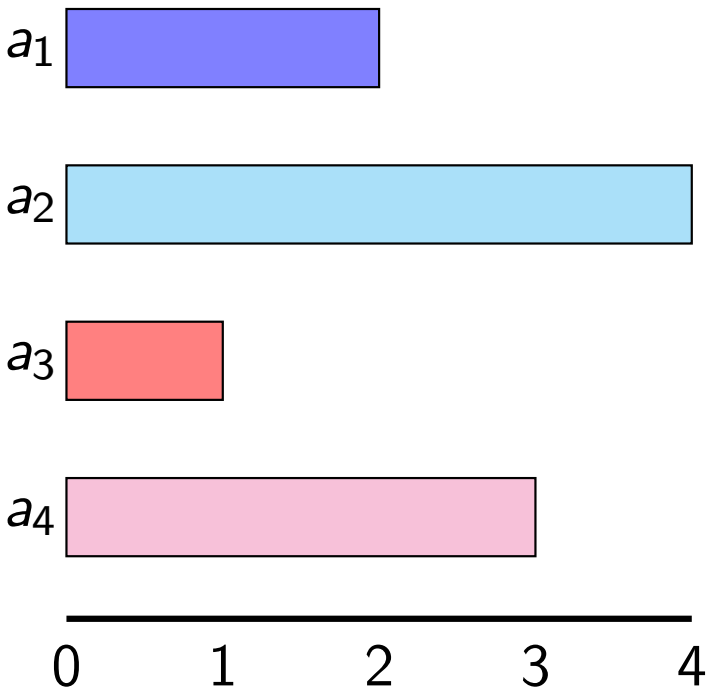

For each item of A, build a gadget with some **key** vertices.

All key vertices must have the same color.

Key vertices with color $i \rightarrow$ item in i-th bin.

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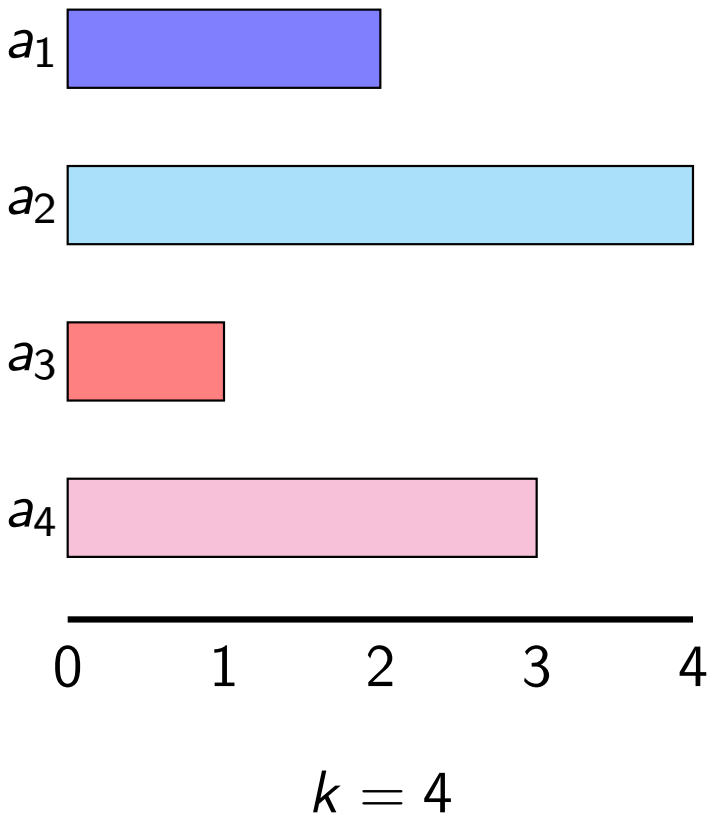

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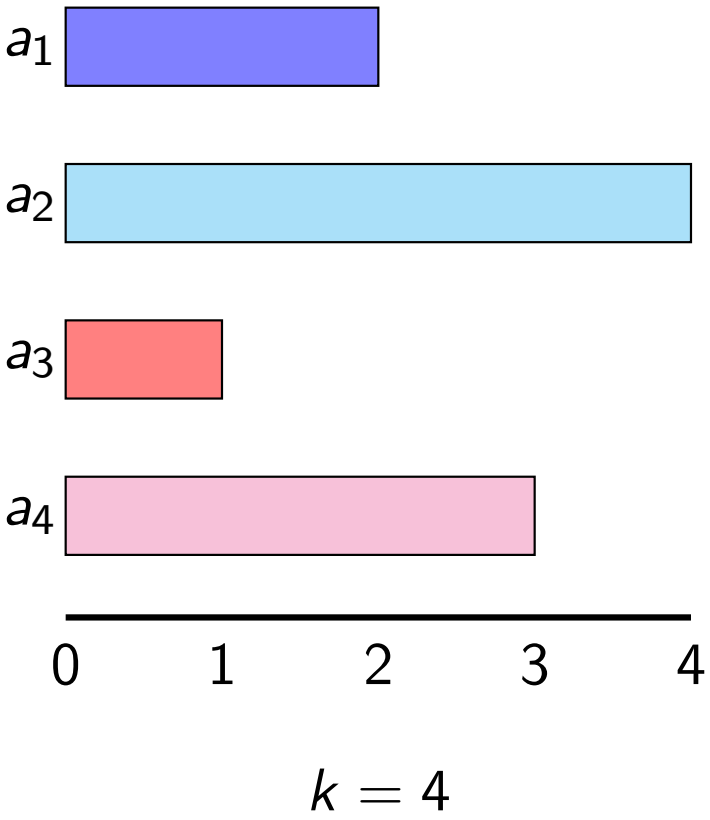
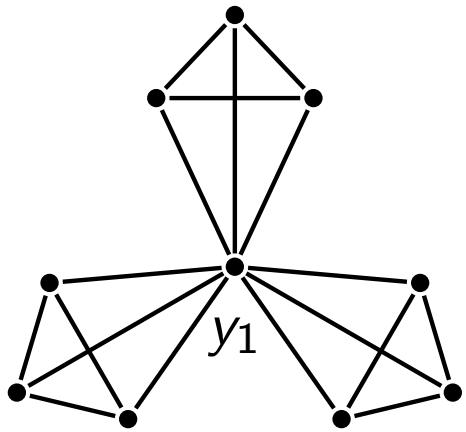
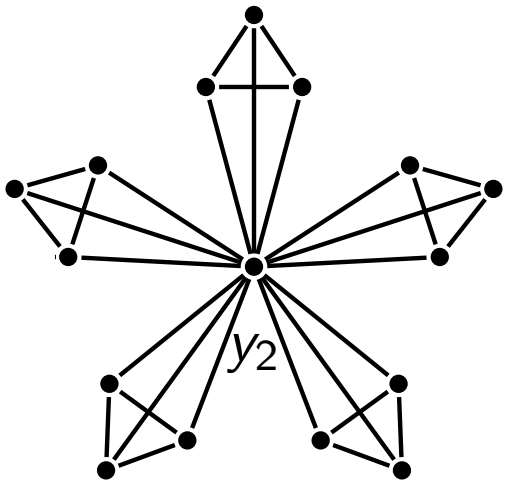
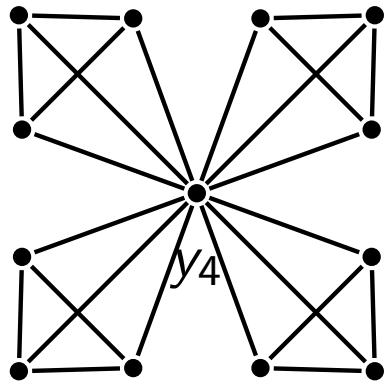
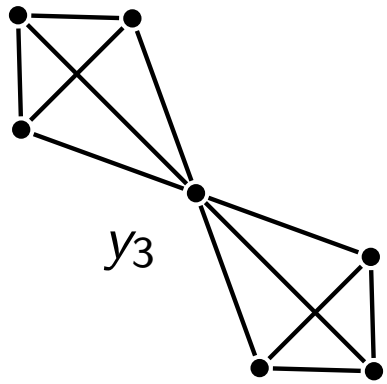


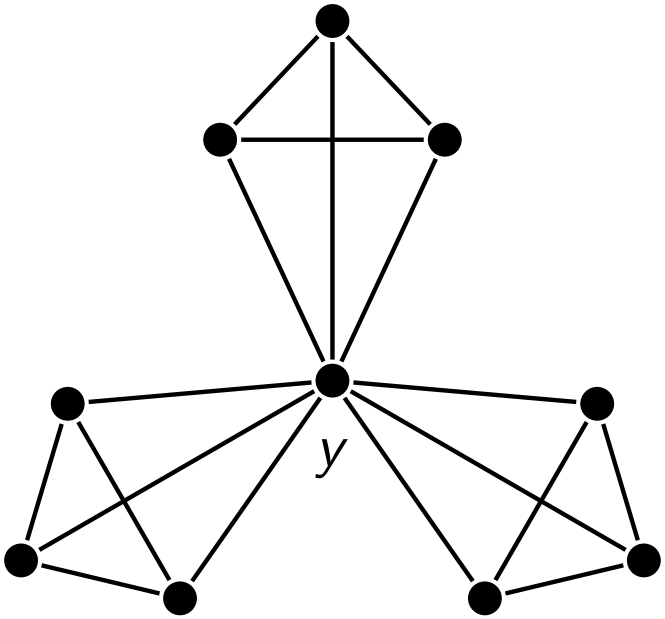




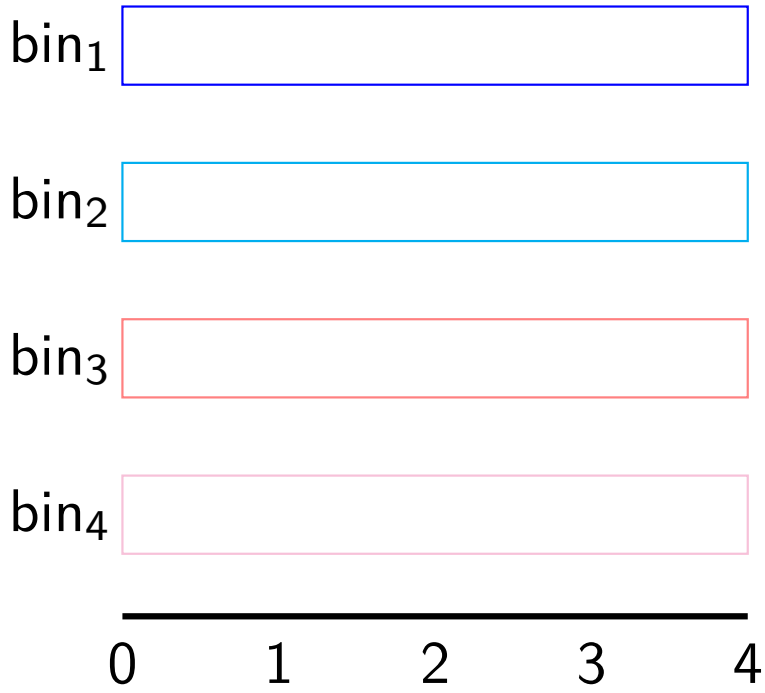




Flowers, colors, and bins



$a_j = 2, k = 4$



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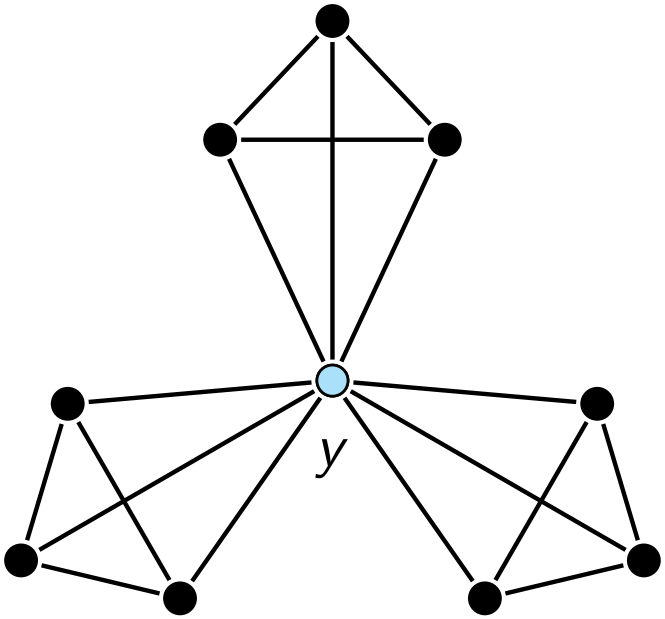

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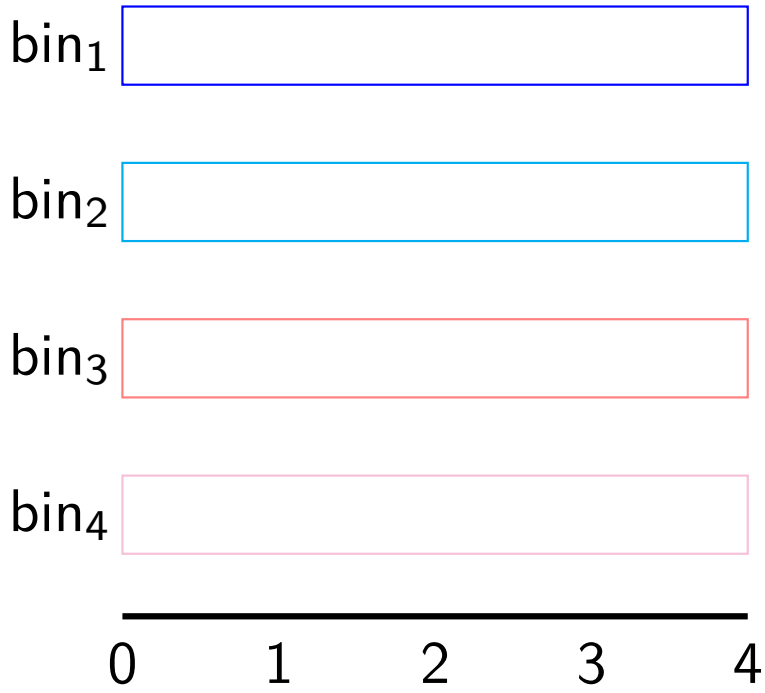

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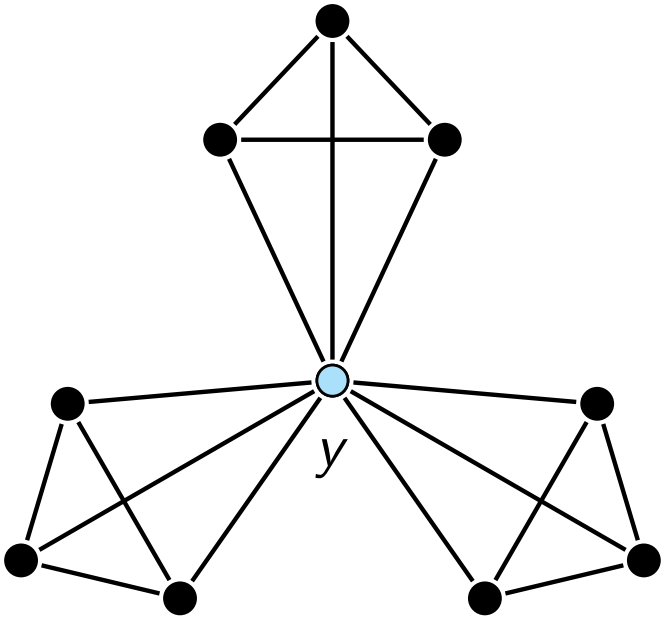

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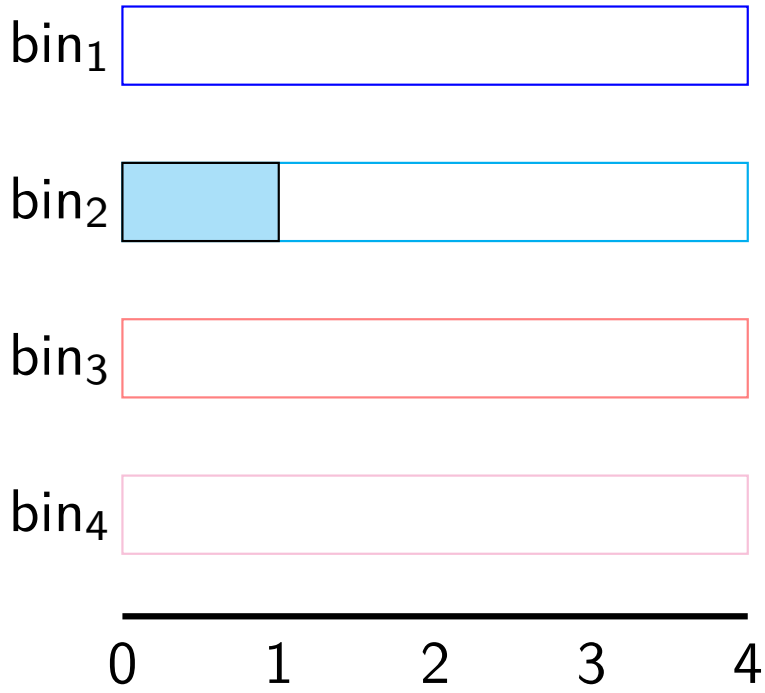

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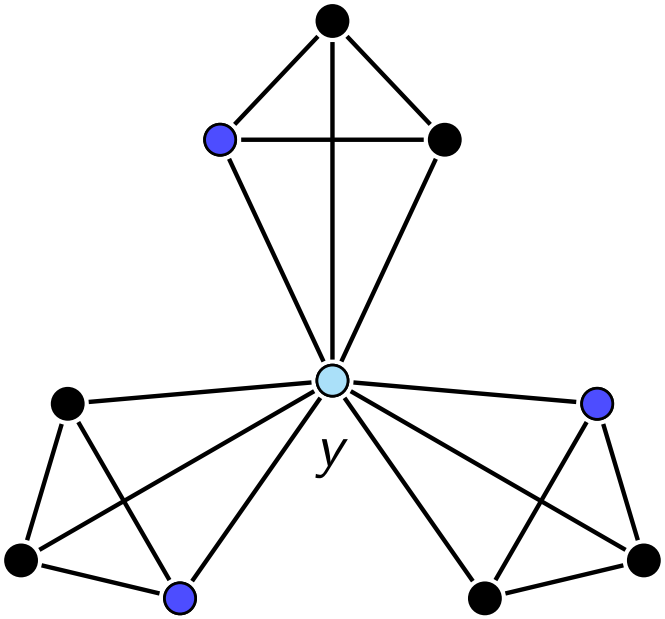

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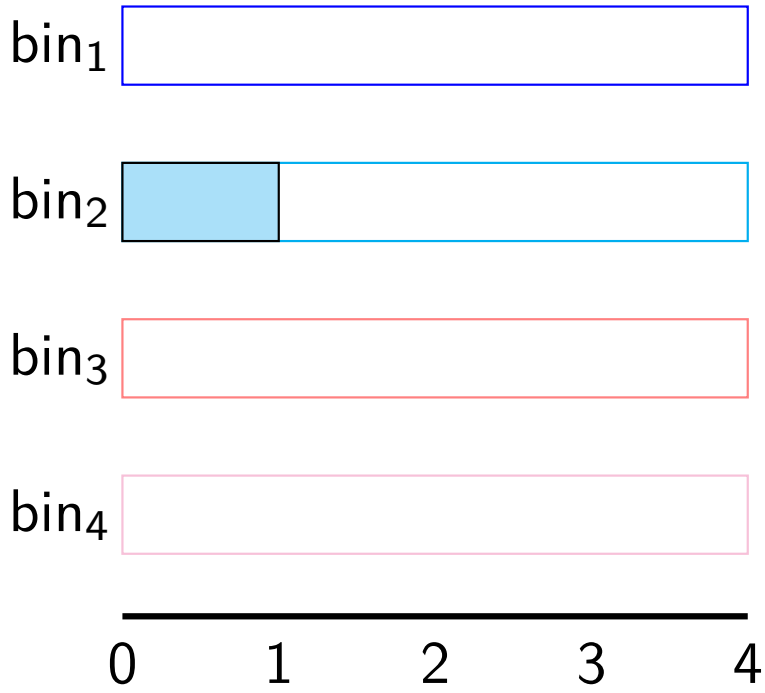

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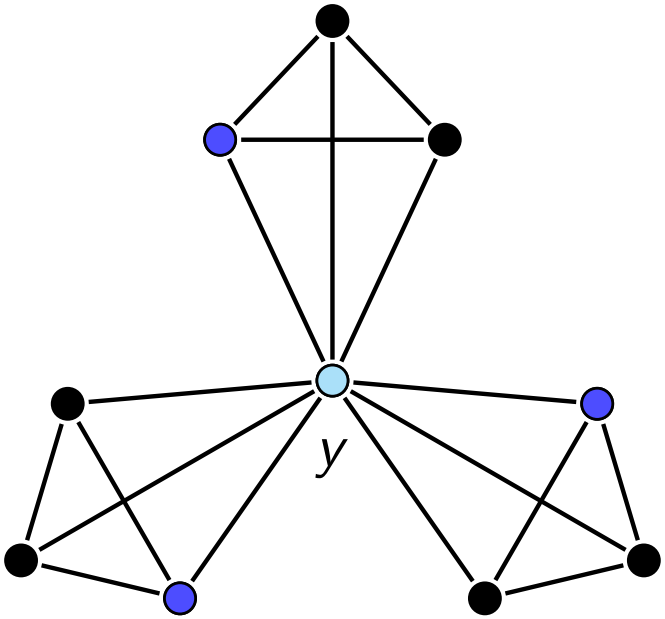

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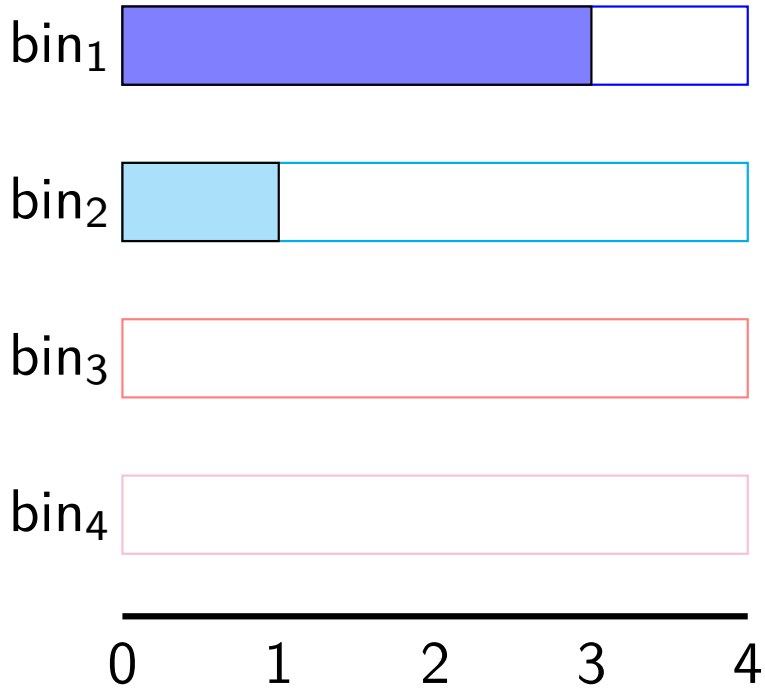

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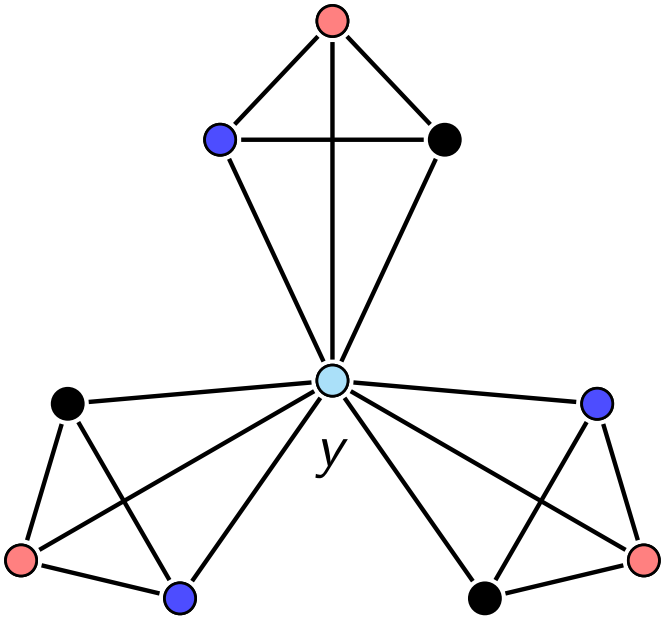

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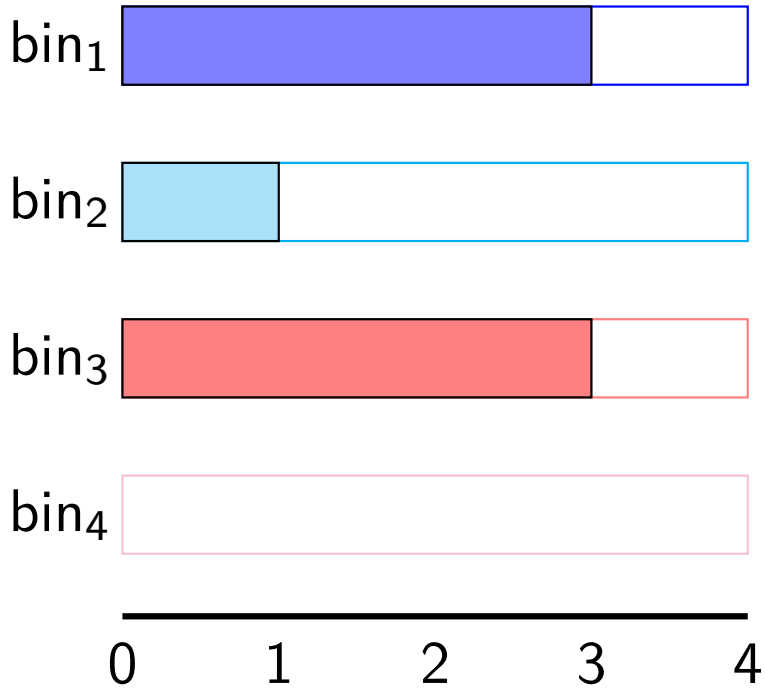

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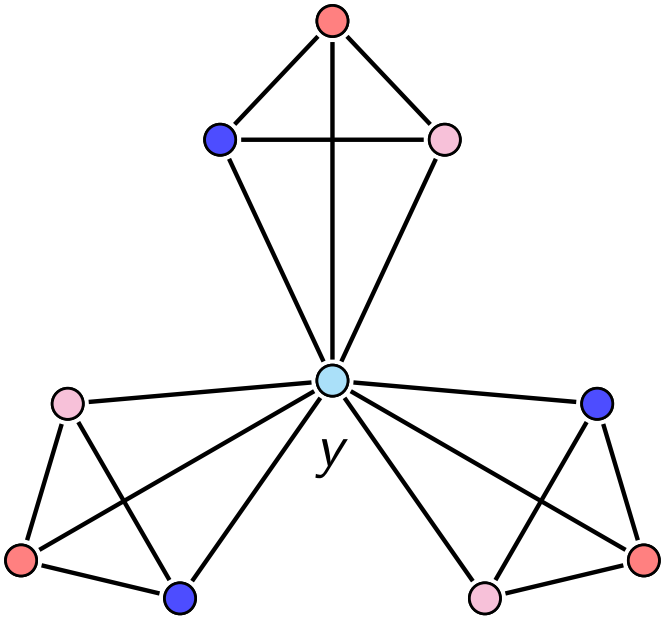

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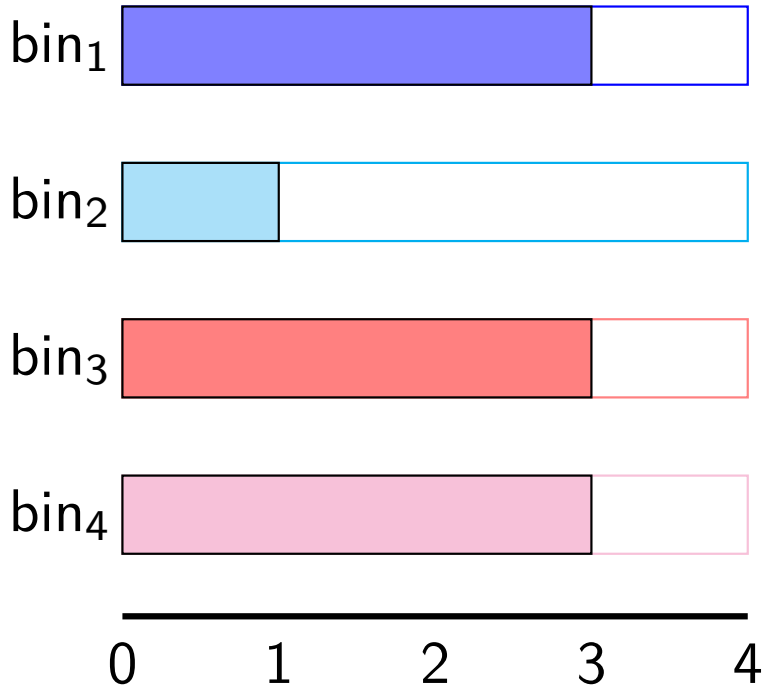

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Block Graphs

Theorem

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B = \sum_{j|y_j \in \psi_i} a_j
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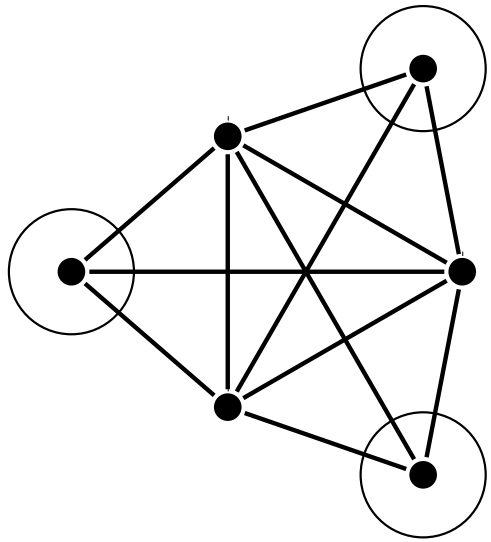

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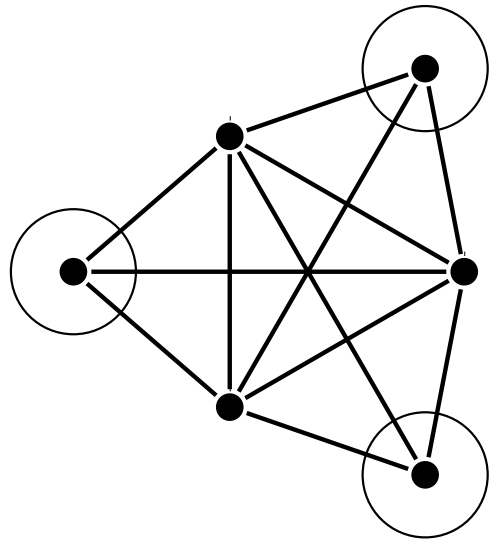

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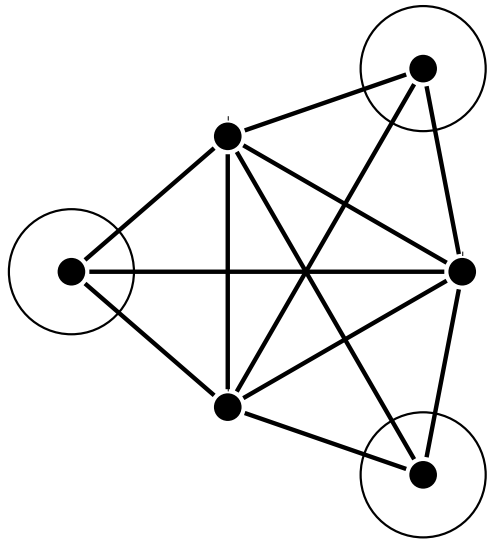
Try equitably k-color it.



$K_{1,r}$-free interval graphs

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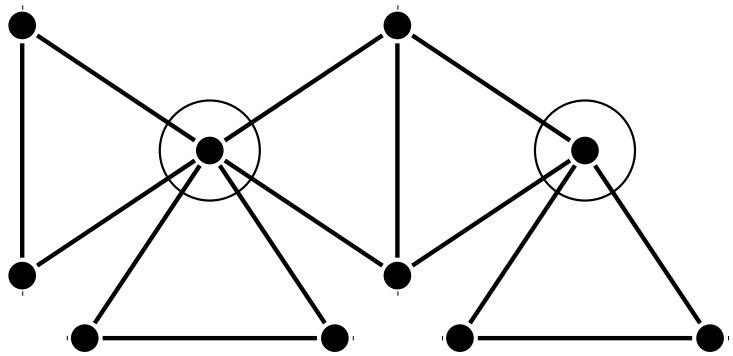

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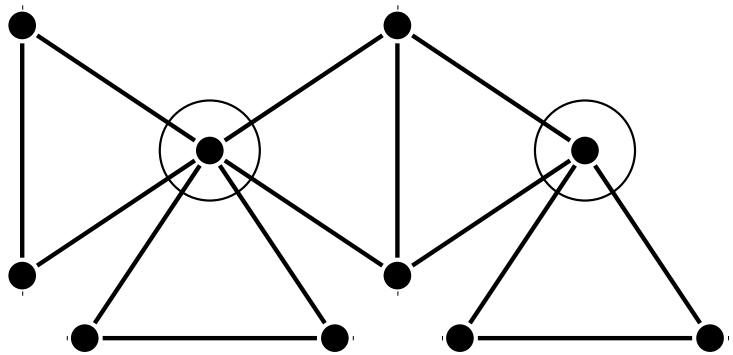

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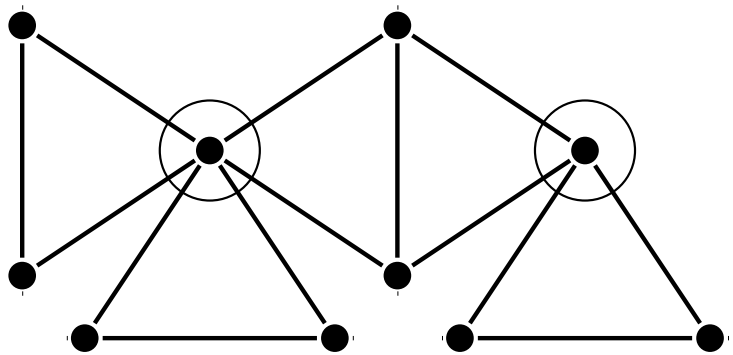

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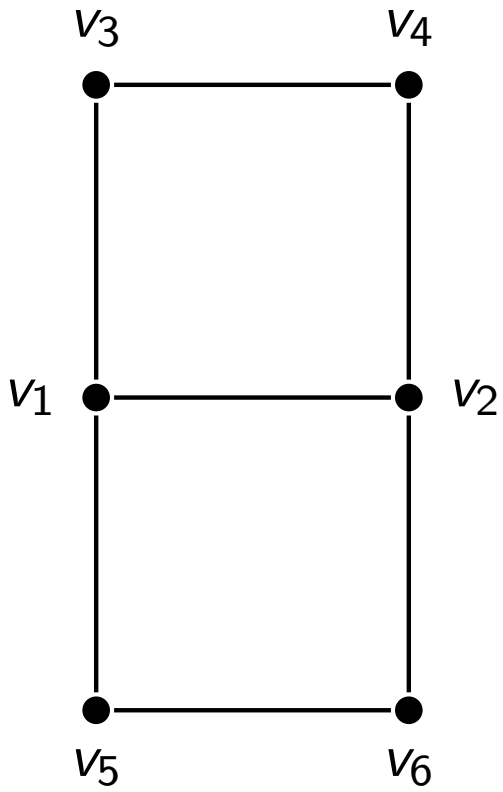

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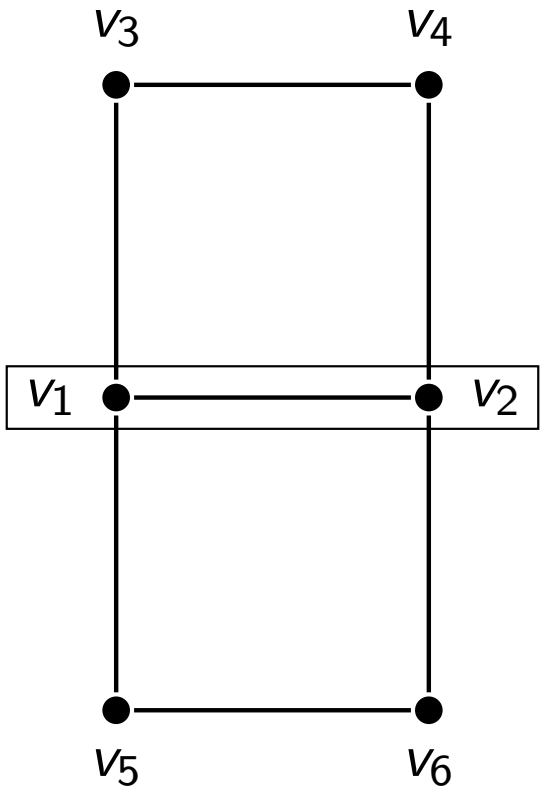
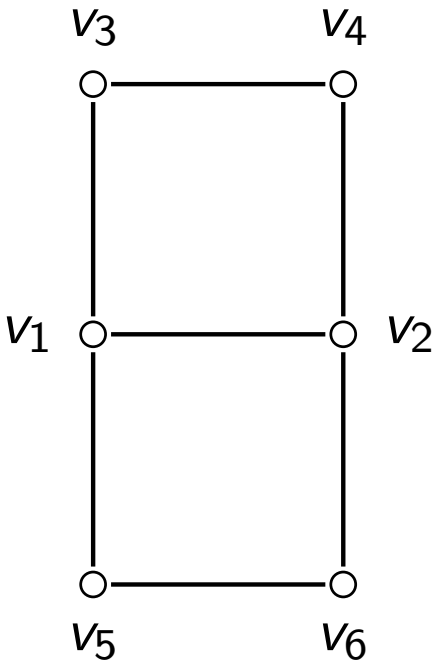


A max-flow based algorithm



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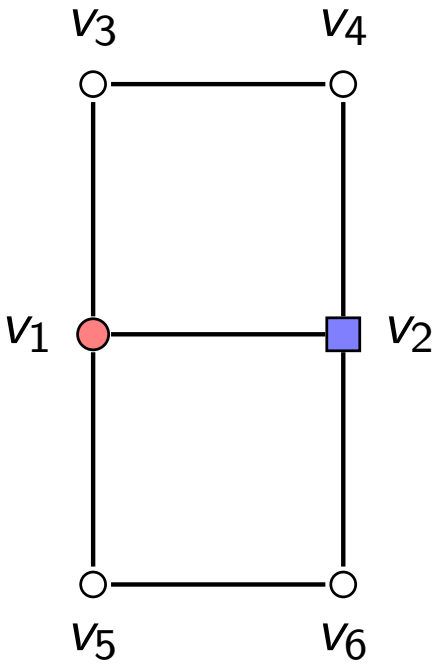
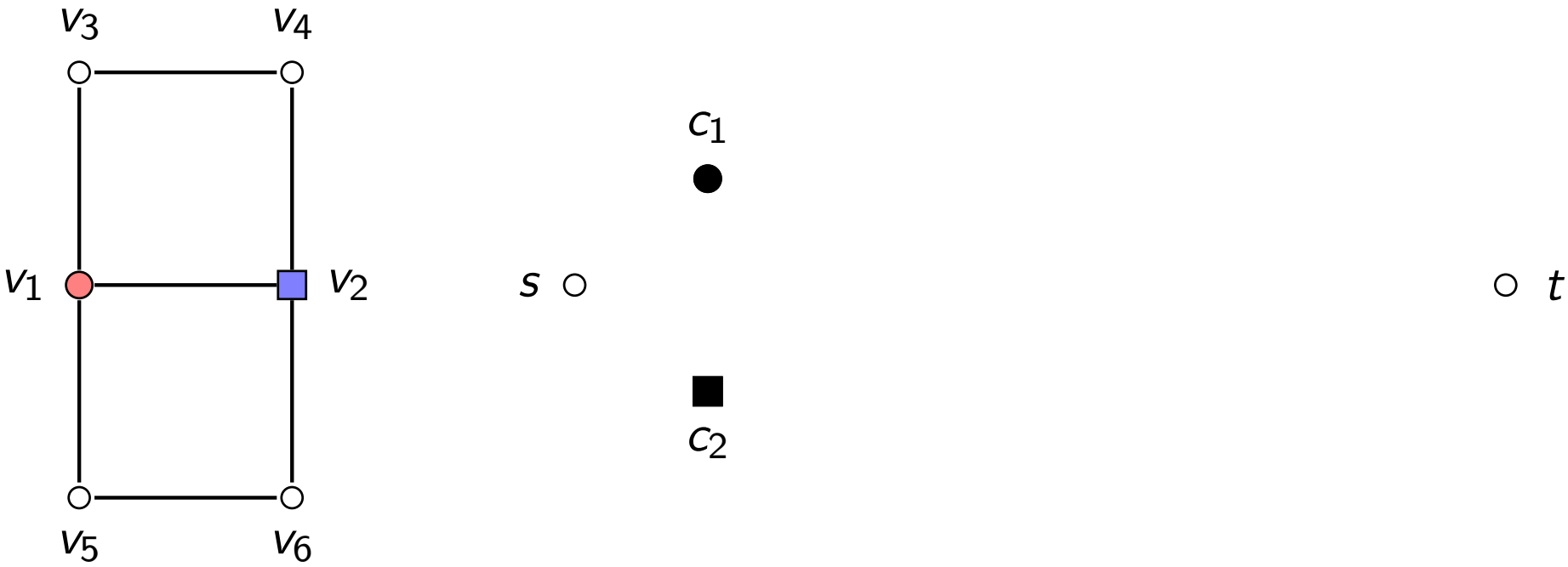


A max-flow based algorithm

Vertices

Source s, sink t,

A max-flow based algorithm



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Source s, sink t, for each color i, c_i,

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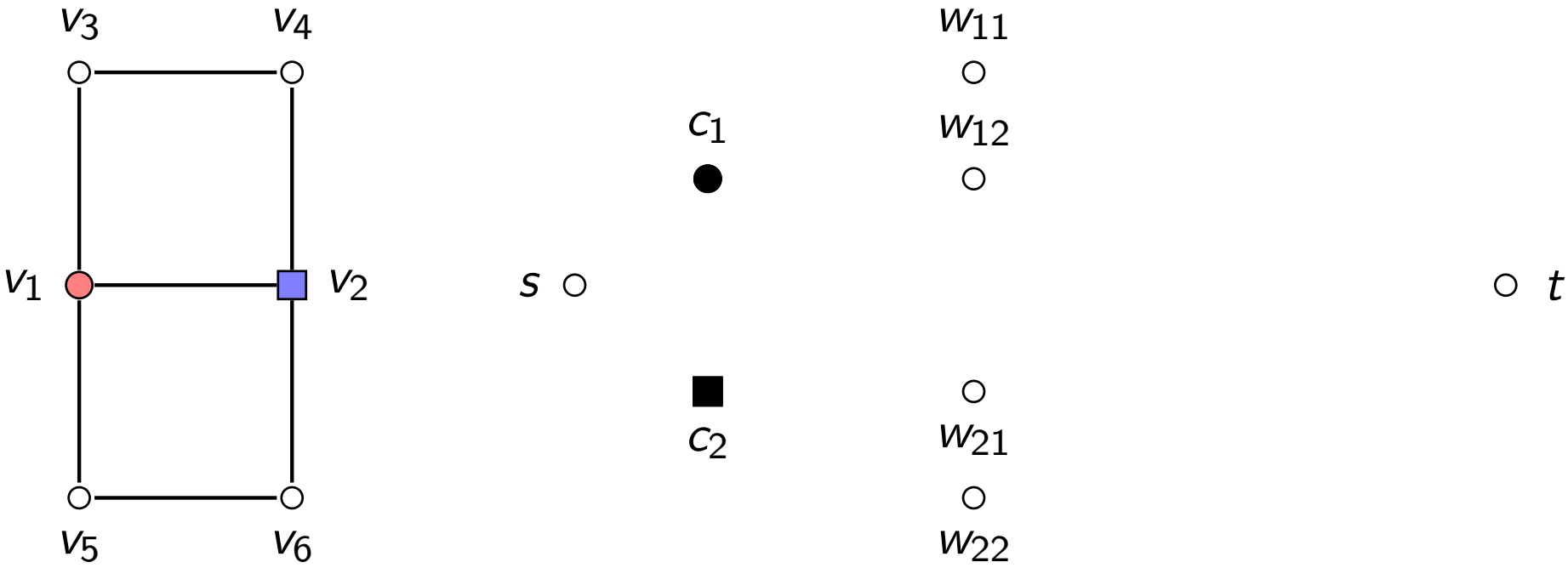

Vertices

Source s, sink t, for each color i, c_i, for each color i and clique j, w_{ij},

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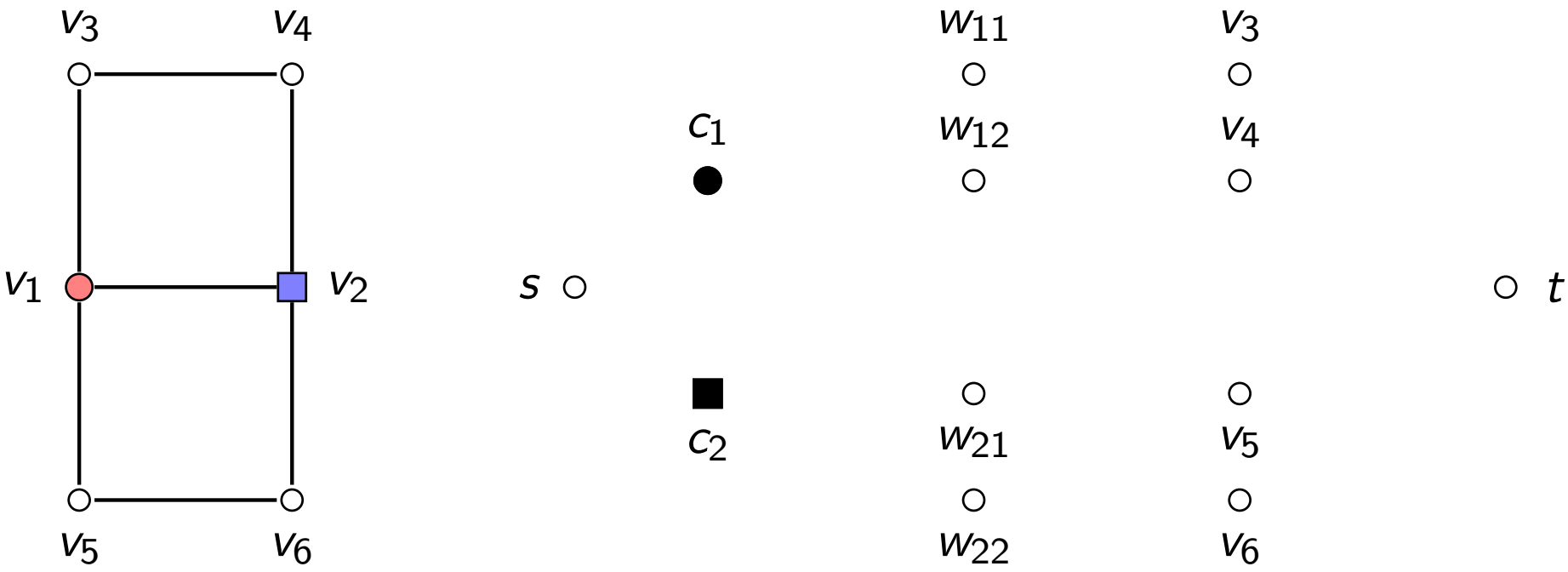

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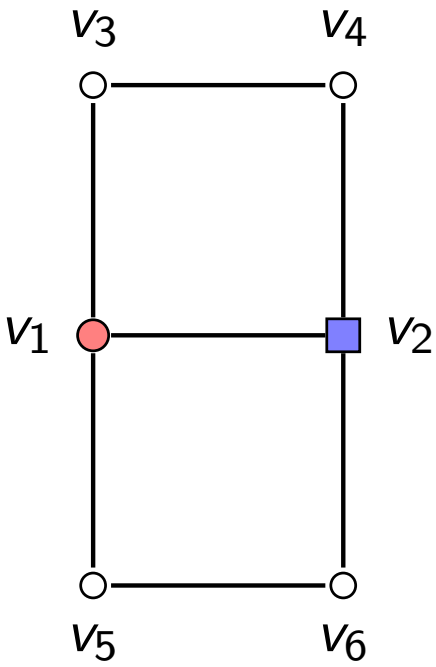

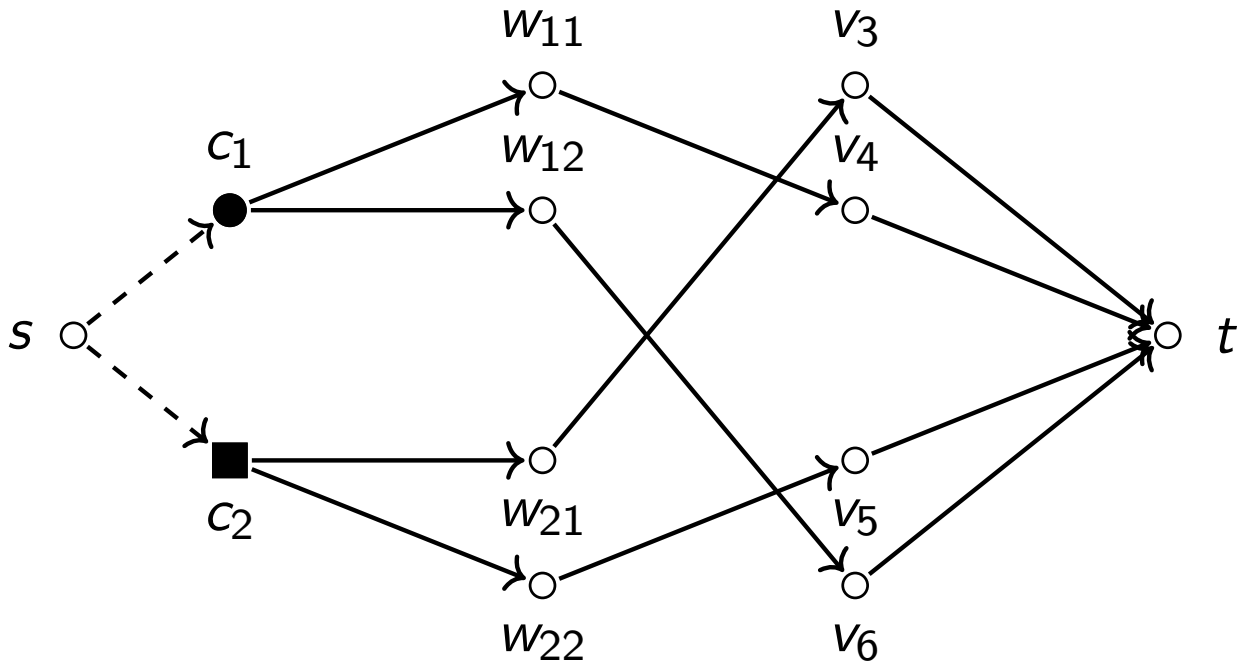

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Each flow unit gives the color of one vertex. Solid arcs have unit capacity.

Parameterized Complexity

A primer on parameterized complexity

If a problem is NP-hard (we believe that) there is no *exact algorithm* that solves *all instances* in time *polynomial* on the size input.

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Kernelization x can be compressed into a *kernel* x' so that $|x'| \leq g(k)$.
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W[1]-hard No $f(k)n^{\mathcal{O}(1)}$ time algorithm unless FPT $=$ W[1].
Example: Clique parameterized by size of the solution.

The parameterized story so far...

Class	Parameterized Complexity
Bipartite	W[1]-hard parameterized by #colors
Cographs	W[1]-hard parameterized by #colors
Chordal	W[1]-hard parameterized by #colors
Block	?
Disjoint union of Split	?
interval	?
Independent set $+kv$	FPT param. by k
Split $+kv$	W[1]-hard parameterized by k
Cluster $+kv$	?
Co-cluster $+kv$	?
Forest $+kv$	W[1]-hard parameterized by $k+$ #colors
Path $+kv$	?
Max leaf ℓ	FPT parameterized by ℓ

After this talk

Class	Parameterized Complexity
Bipartite	W[1]-hard parameterized by #colors
Cographs	W[1]-hard param. by #colors
Chordal	W[1]-hard param. by #colors
Block	W[1]-hard param. by #colors + treedepth
Disjoint union of Split	W[1]-hard param. by #colors + tw
Interval	W[1]-hard param. by #colors + bandwidth
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Max leaf ℓ	Cubic kernel parameterized by ℓ

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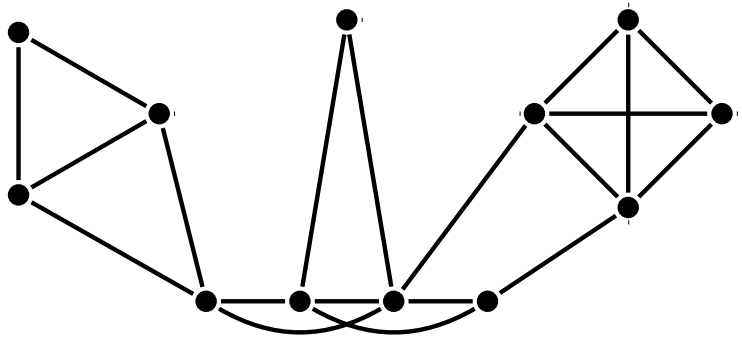

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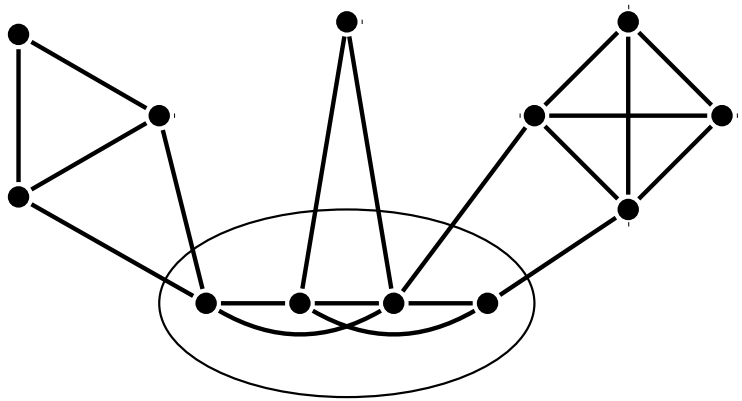

G is a cluster $+kv$ graph if there is a set $U \subset V(G)$ of size k such that $G - U$ is a cluster graph, with clusters $\{C_1, \ldots, C_\ell\}$.

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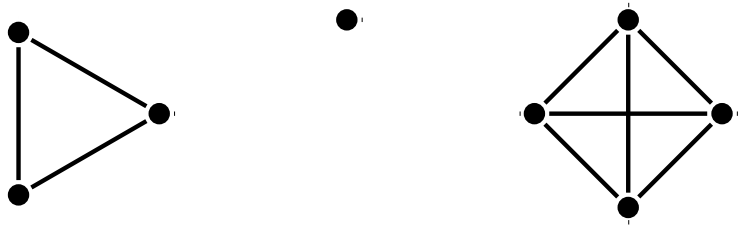

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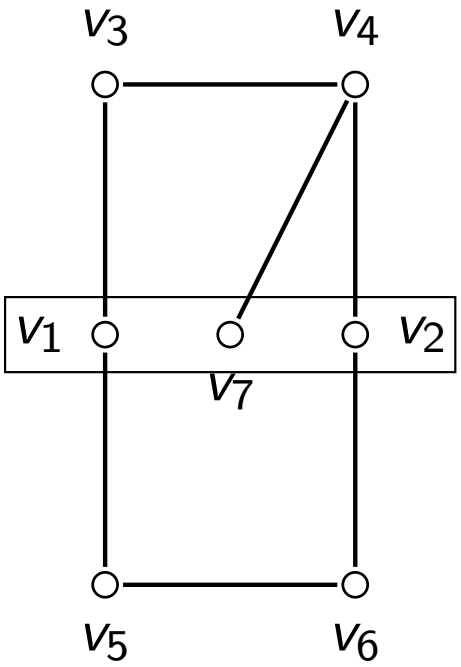
Cluster $+kv$: max-flow again



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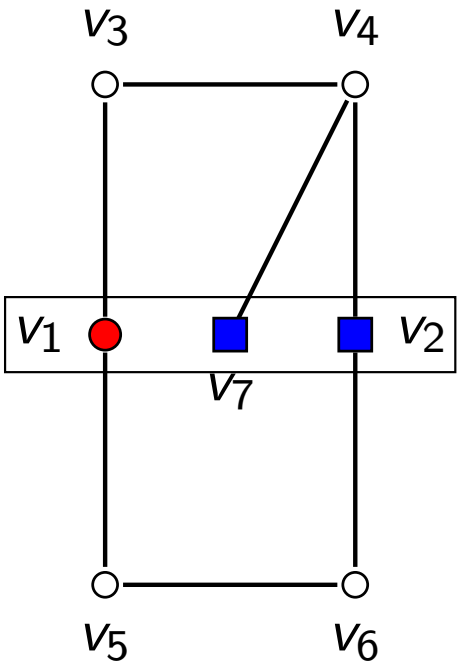

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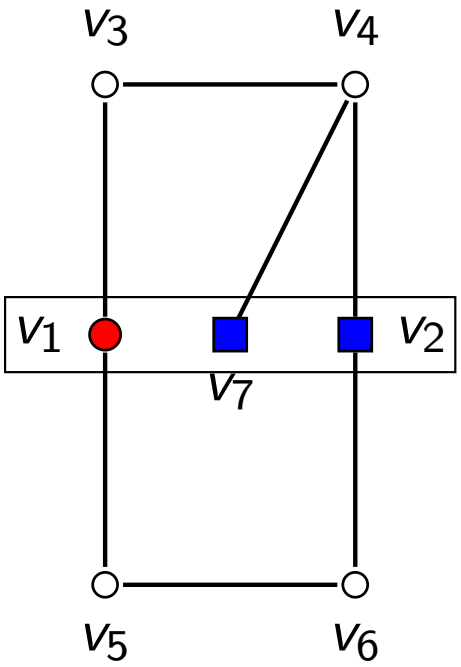

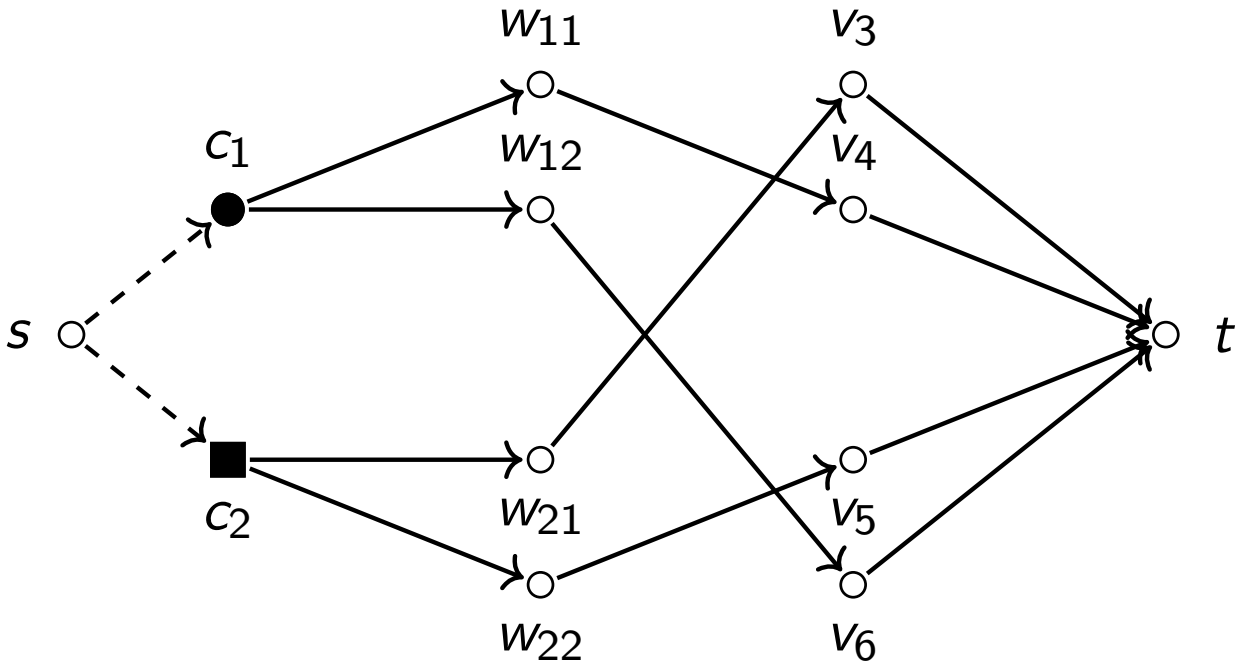

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For each of the k^k colorings of U, construct the auxiliary graph.

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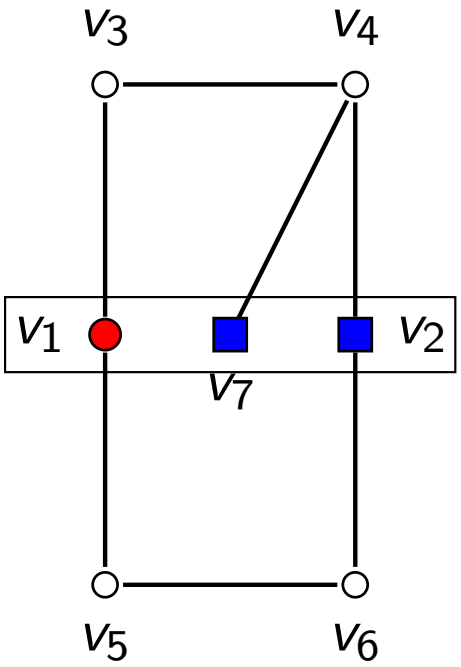

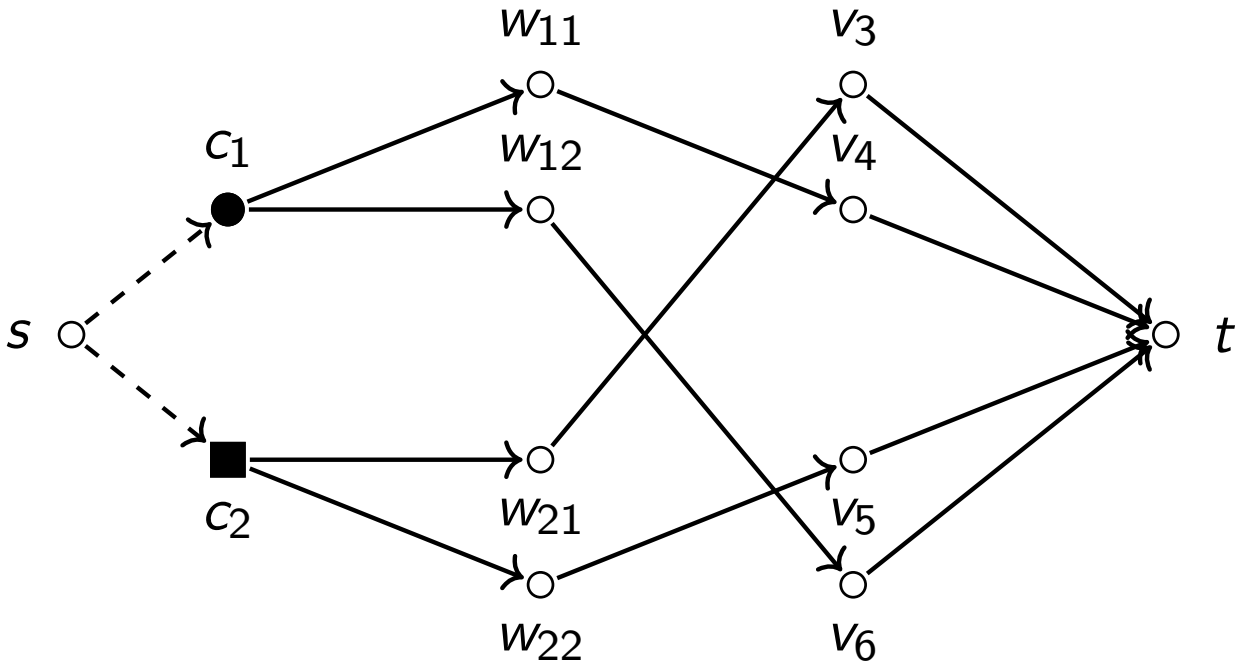

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The max leaf number ℓ of a graph G is the maximum number of leaves on a spanning tree of G.

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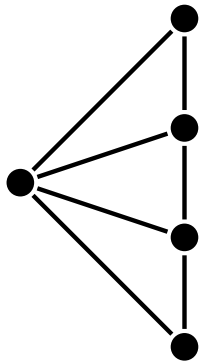

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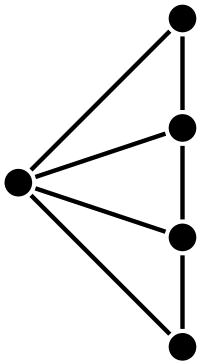

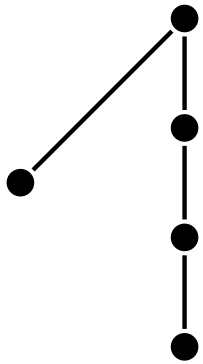

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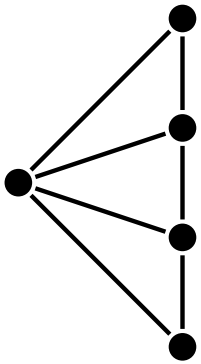

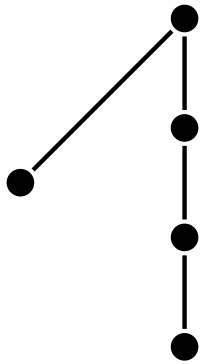

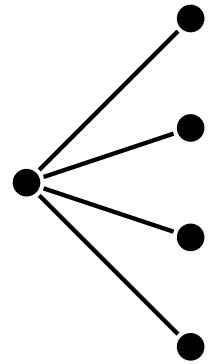

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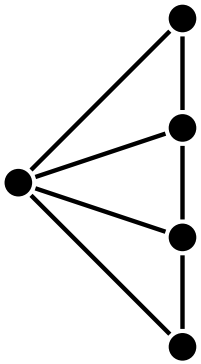
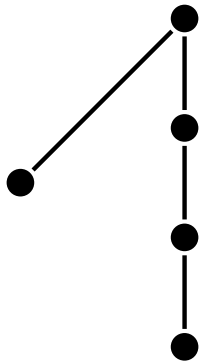
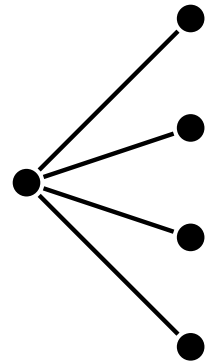

Theorem (Estivill-Castro et al. (2005))

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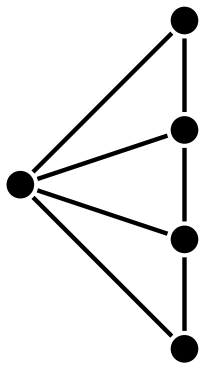
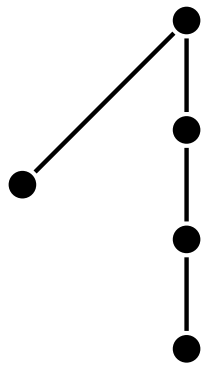
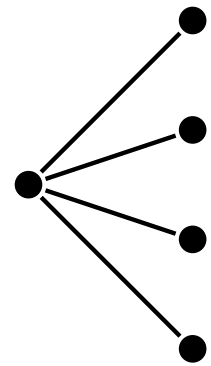

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Theorem (Cappelle, G., dos Santos (LAGOS2021?))

$G \setminus H$ is a collection of at most 5.5ℓ paths and contains most vertices of G.

Improving the balance of the force/coloring

Let φ be a k-coloring of H and $\alpha = |\varphi_k| - |\varphi_1|$.

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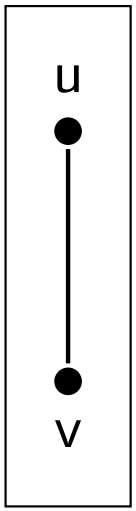
$V(H)$



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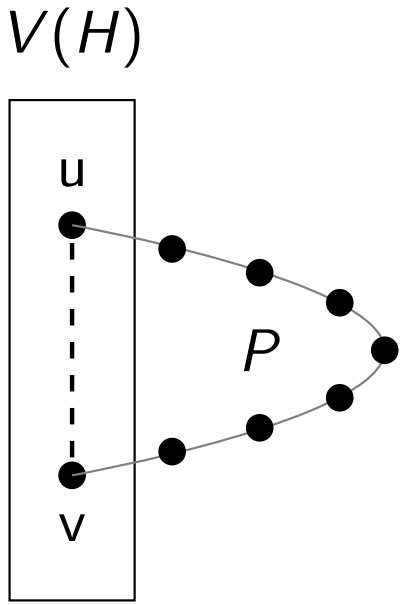

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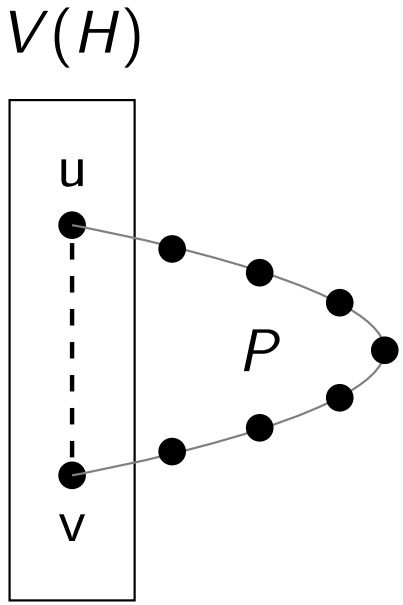

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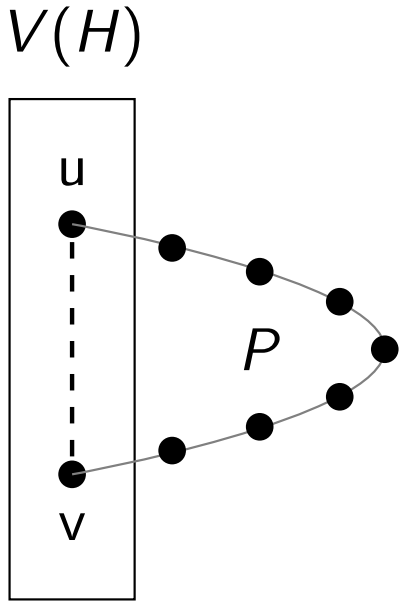

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If $|P| \geq k + 2 + x(k + 1)$, we can extend φ to a coloring that is $\alpha - x$ or 1-balanced.

A kernel parameterized by k and ℓ

Let Q be the subgraph of G obtained by removing all paths with $\geq 2k+3$ vertices.

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Observation 3

Otherwise, the paths in $G \setminus Q$ satisfy $\sum_{P_i \in G \setminus Q} x_i(k+1) \geq |Q|(k+1)$, so we can bring any coloring of Q to G and make it equitable.

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Replacing on Observation 2: G has at most $176\ell^3 + \mathcal{O}\left(\ell^2\right)$ vertices.

Back to the future: Vertex Cover

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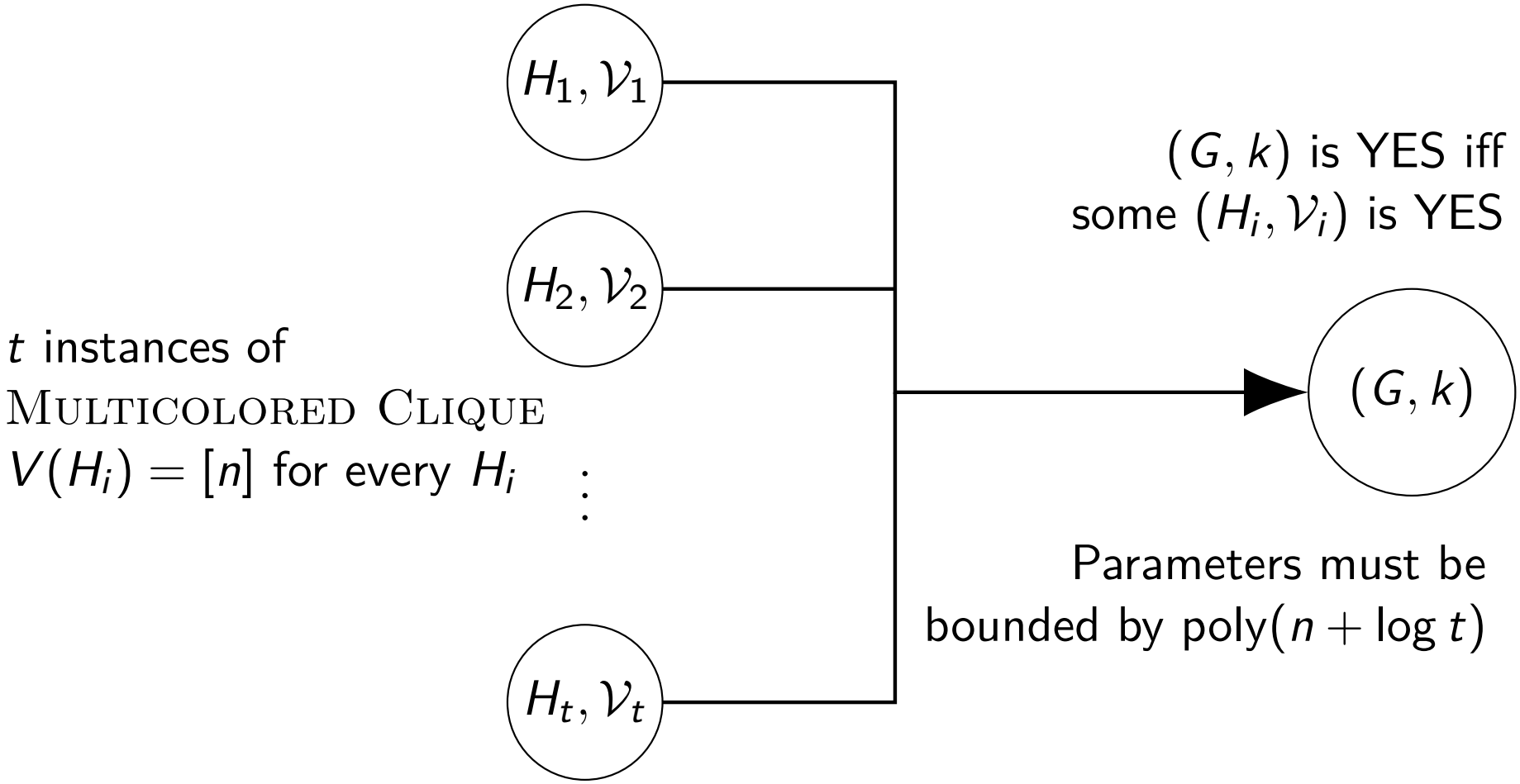
But lets try to get some very high level intuition.

OR-cross-composition



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(G, k)

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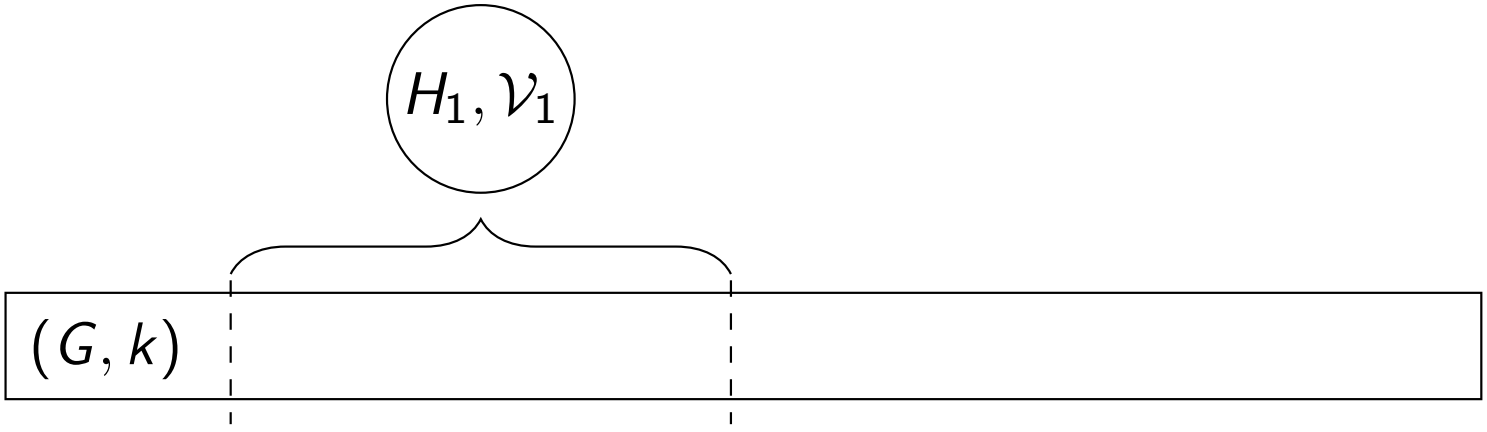

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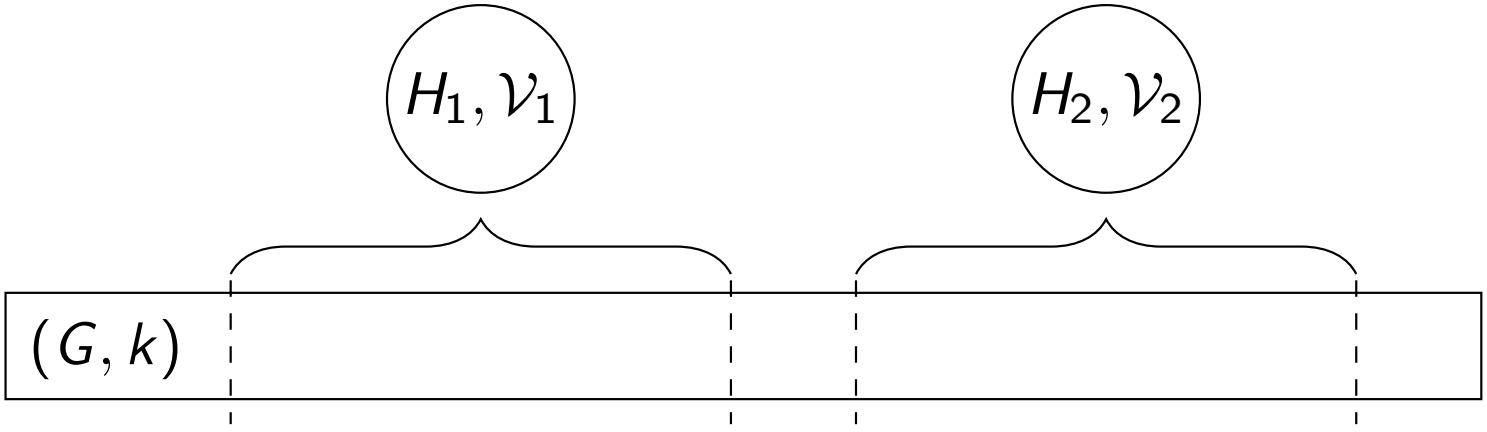

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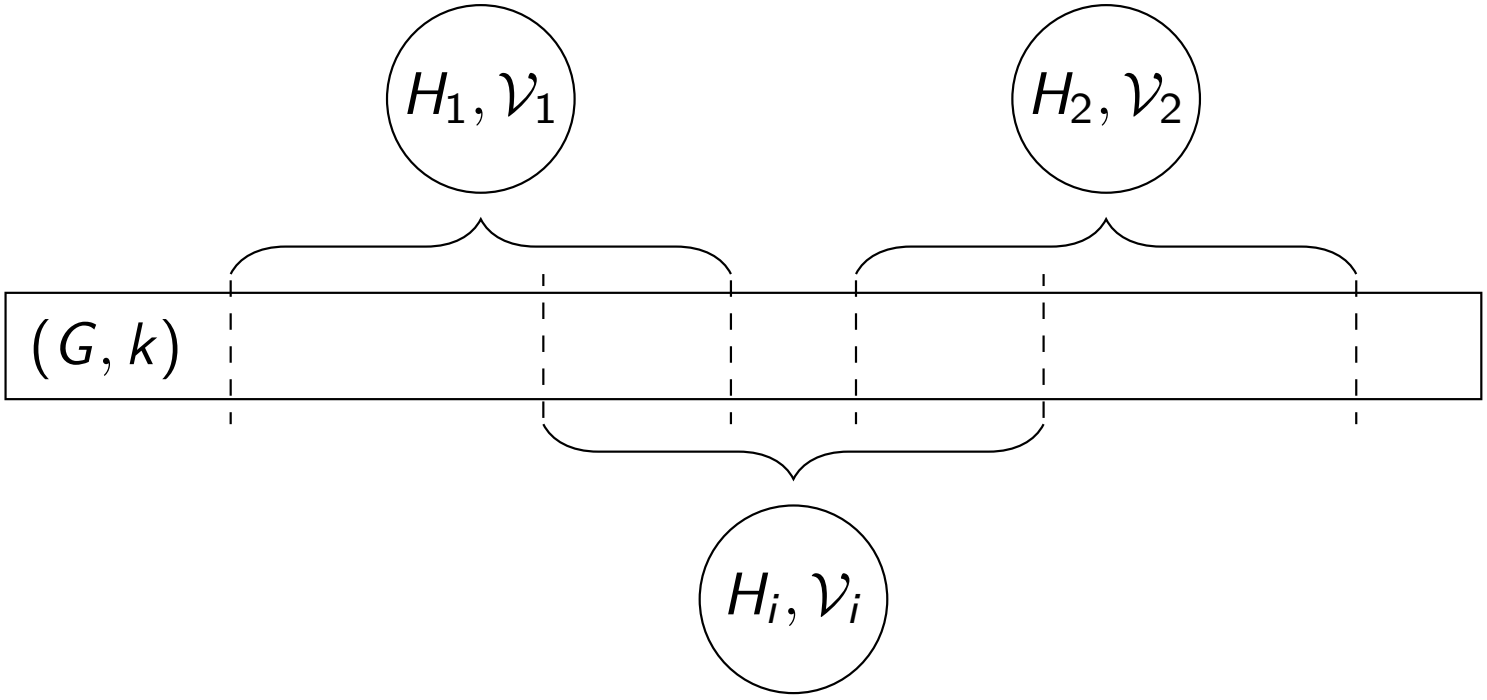

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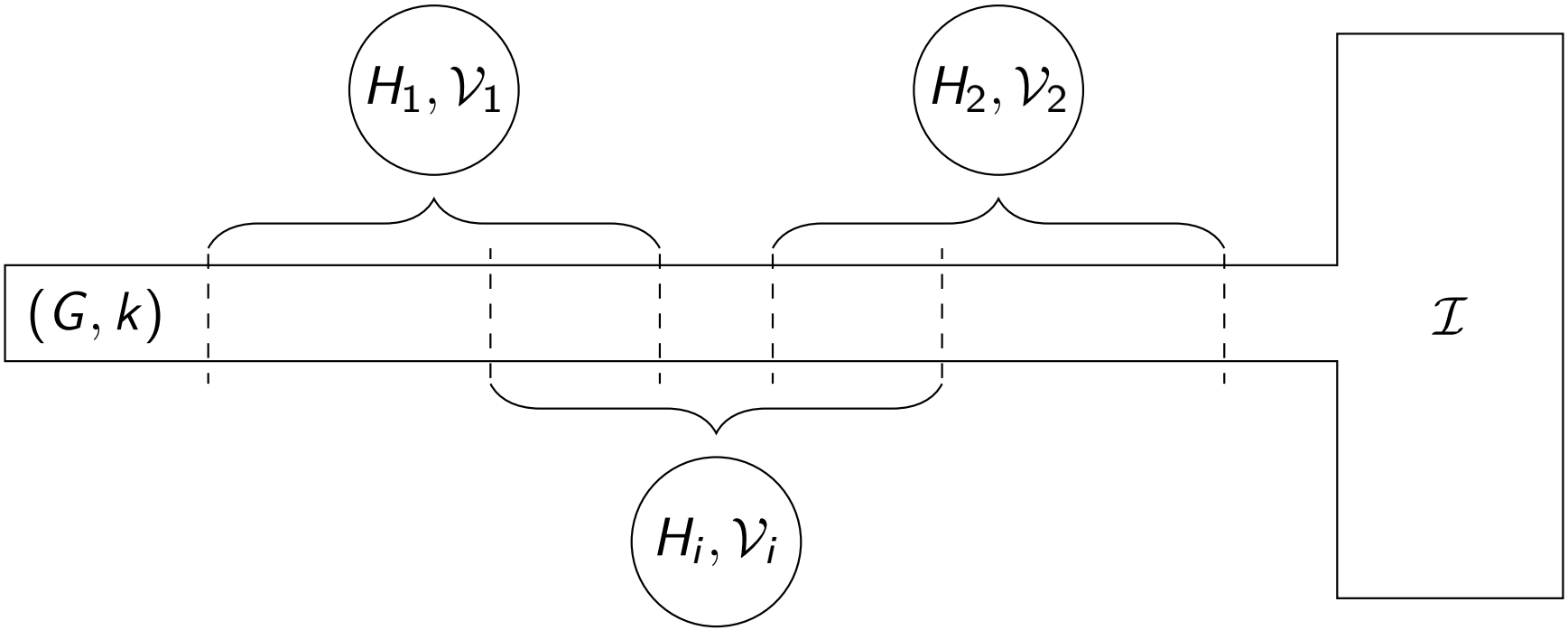

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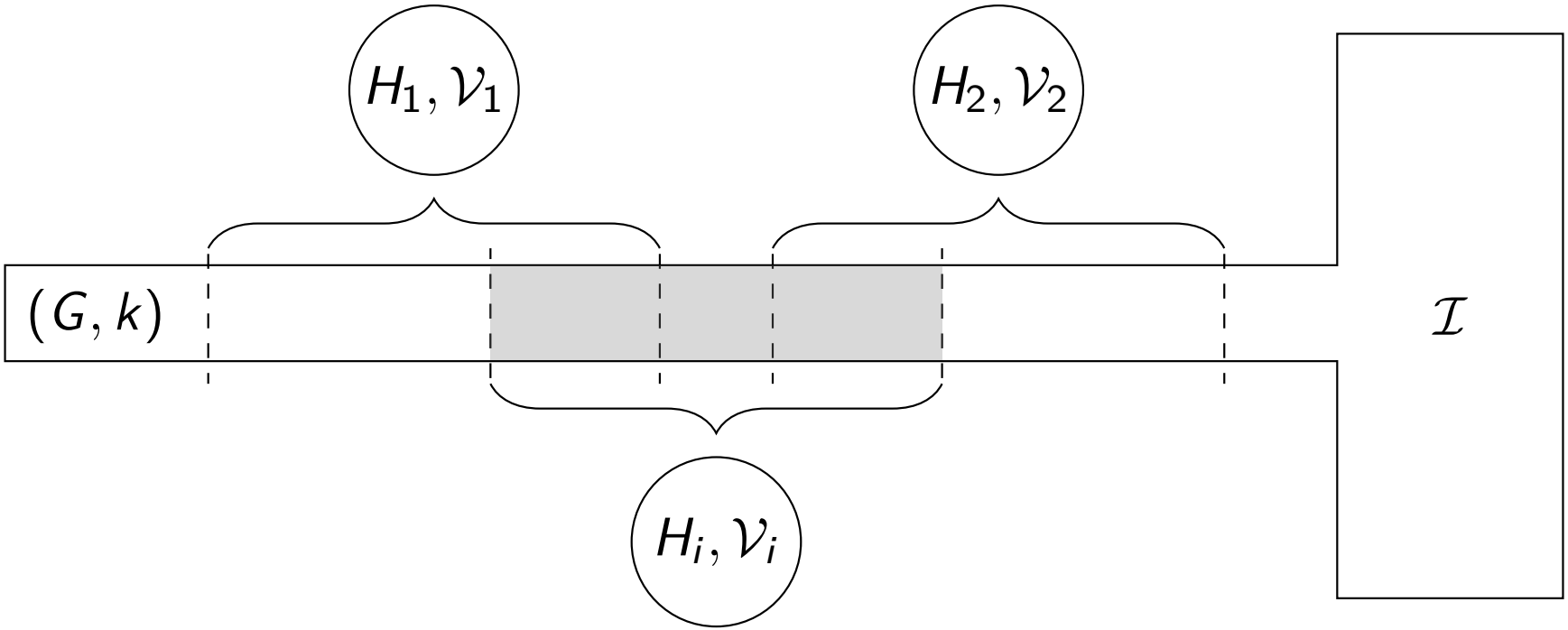

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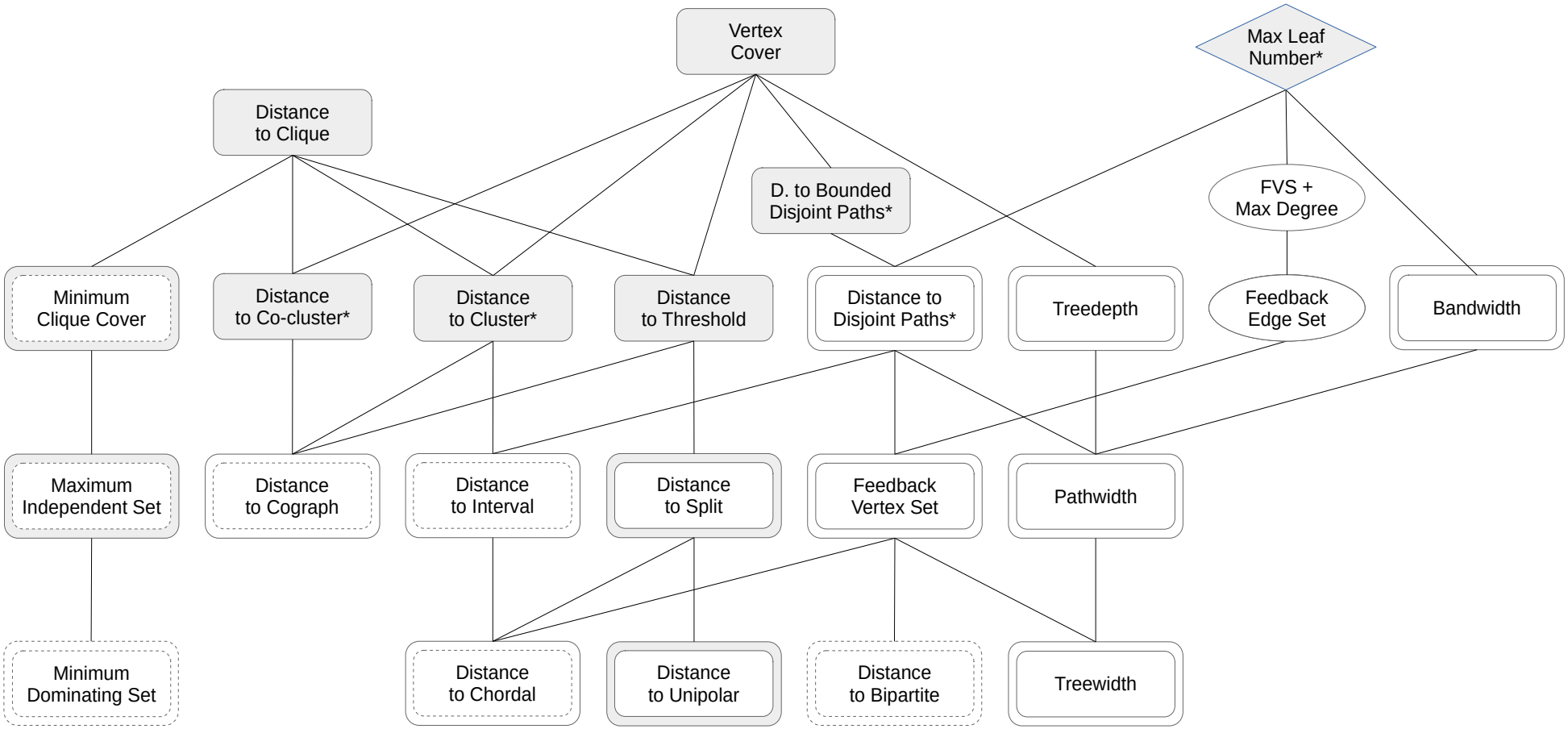
Parameterized landscape



Open Problems

- Can we find a subcubic kernel when parameterizing by the max leaf number?
- Using ETH, can we prove that currently known algorithms have optimal running time?
- Is Equitable Coloring FPT when parameterized by feedback edge set?

Thank you!