On the Δ-interval and Δ-convexity numbers of graphs and graph products

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Discrete structures

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- A family $\mathcal{C}$ of subsets of a finite set X is a **convexity on** X if

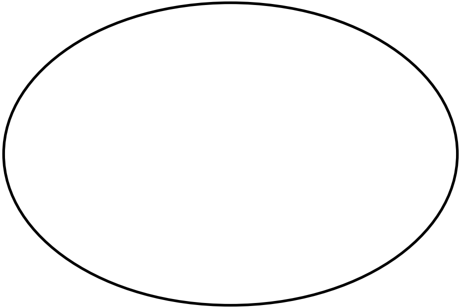
Discrete structures

- A family $\mathcal{C}$ of subsets of a finite set X is a **convexity on** X if
  - $\emptyset, X \in \mathcal{C}$
  - $\mathcal{C}$ is closed under intersections

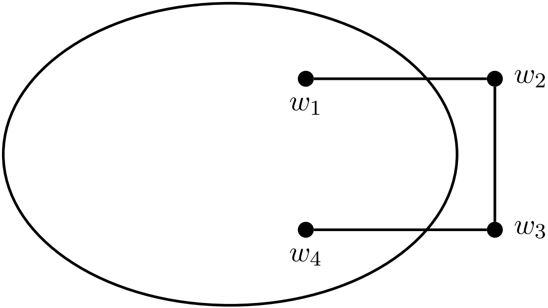
Convex set in $\mathbb{R}^n$

A set is **convex** if it contains the line segment joining any two vertices of the set

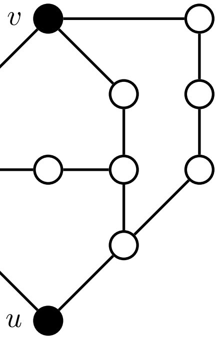
Path convexities



Path convexities



Geodetic convexity



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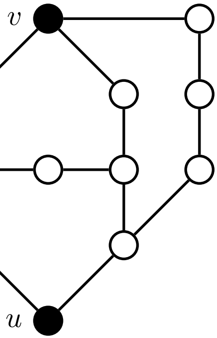

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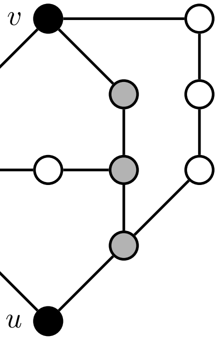

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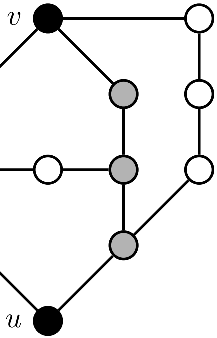

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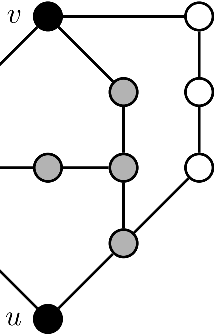

Geodetic convexity

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- S is **convex** if S is equal to its interval
- **Convex hull of** S is the minimum convex set containing S

Graph convexities

Most graph convexities use an interval funcion on a family of paths

Graph convexities

- Shortest paths (**Geodetic convexity**)

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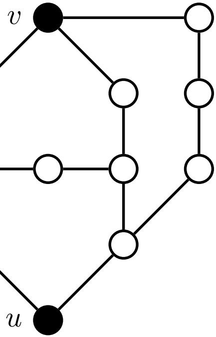
Graph convexities

- Shortest paths (**Geodetic convexity**)
- Induced paths (**Monophonic convexity**)
- Paths of length two (P_3-**convexity**)
- Induced paths of length two (P_3^*-**convexity**)
- Paths of length two forming a triangle (Δ-**convexity**)

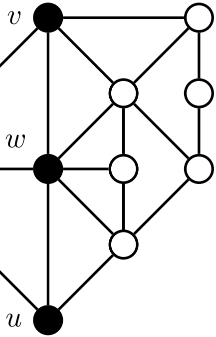
Motivation

- Opinion propagation

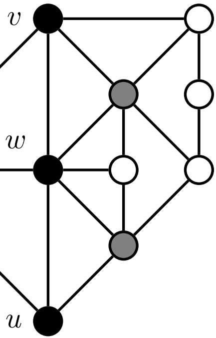
Δ-convexity



Δ-convexity



Δ-convexity



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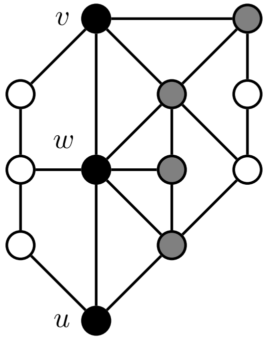

Main problems

- Carathéodory number
- Radon number
- Helly number

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- Carathéodory number
- Radon number
- Helly number
- Hull number
- Convexity number
- Interval number

Problems

- **Convexity number** is the maximum cardinality of a Δ-convex set properly contained in $V(G)$

Problems

- **Convexity number** is the maximum cardinality of a Δ-convex set properly contained in $V(G)$
- **Interval number** is the minimum cardinality of a set whose Δ-interval is $V(G)$

Convexity number

Theorem

The MAX CONVEX SET *problem is* W[1]*-hard and* NP*-hard to approximate within a factor* $O(n^{1-\varepsilon})$ *for any constant* $\varepsilon > 0$ *in the* Δ*-convexity even for graphs with diameter* 2.

Convexity number

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Theorem

For a chordal graph G, $c_\Delta(G)$ can be computed in $O(nm)$ steps.

Graph products

- **Cartesian product** $G \Box H$: two vertices are adjacent if they are adjacent in one coordinate and equal in the other

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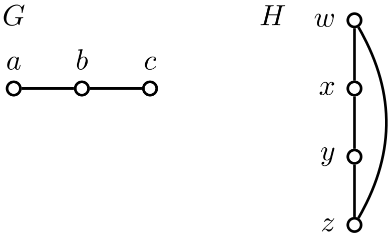
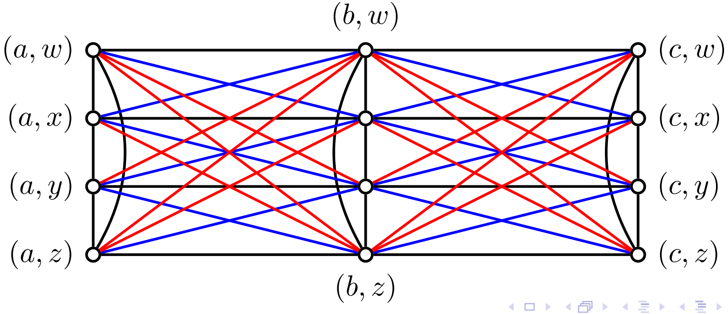
- **Lexicrographic product** $G \circ H$: (adjacent in the first coordinate) or (equal in the first coordinate and adjacent in the second)

Graph products





Definitions

- Δ-extreme vertex

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- $\gamma(G)$: minimum dominating set of G

Results

Theorem

Let G and H be connected graphs having no Δ-extreme vertices with orders n and m, respectively. Then, $c_{\Delta}(G \Box H) = mn' + nm' - n'm'$ where $n' = c_{\Delta}(G)$ and $m' = c_{\Delta}(H)$.

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Let G and H be connected non-trivial graphs. Then, every two adjacent vertices form a Δ-hull set of $G * H$ for $* \in \{\boxtimes, \circ\}$.

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Let G and H be connected non-trivial graphs. Then, every two adjacent vertices form a Δ-hull set of $G * H$ for $* \in \{\boxtimes, \circ\}$.

Corollary

Let G and H be two non-trivial graphs, then
a) $c_\Delta(G \boxtimes H) = \alpha(G \boxtimes H)$, *and*
b) $c_\Delta(G \circ H) = \alpha(G \circ H)$.

Interval number

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The Δ-INTERVAL NUMBER *problem is* NP*-complete for general graphs.*

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Theorem

For a block graph G, $i_\Delta(G)$ *can be computed in* $O(n^2)$ *steps.*

Proposition

If G *is a block graph that is not a complete graph, then* $n_p + 1 \leq i_\Delta(G) \leq n_c + n_p$, *where* n_c *is the number of cut vertices of* G *and* n_p *is the number of pendant blocks of* G.

Interval number

Proposition

Let G and H be non-trivial graphs. Denote by n, n' and n'' the order of G, $i_\Delta(G)$ and the numbers of Δ-extreme vertices in G, respectively. Analgously, define m, m' and m'' for H. Then, $i_\Delta(G \Box H) \leq \min\{n(m' - m'') + m''n', m(n' - n'') + n''m'\}$.

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Proposition

For connected graphs G and H, $2 \leq i_\Delta(G \circ H) \leq 2 i_\Delta(G)$.

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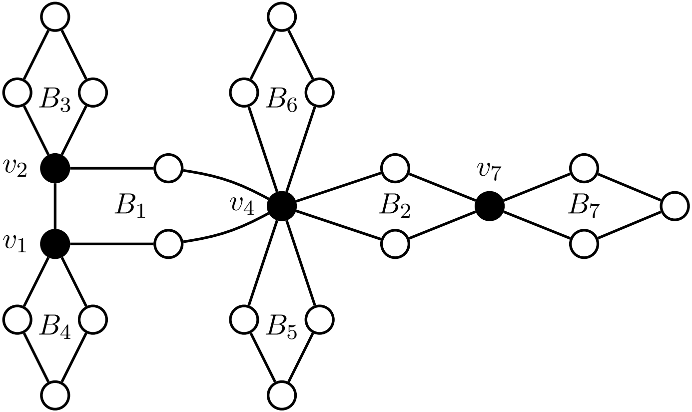
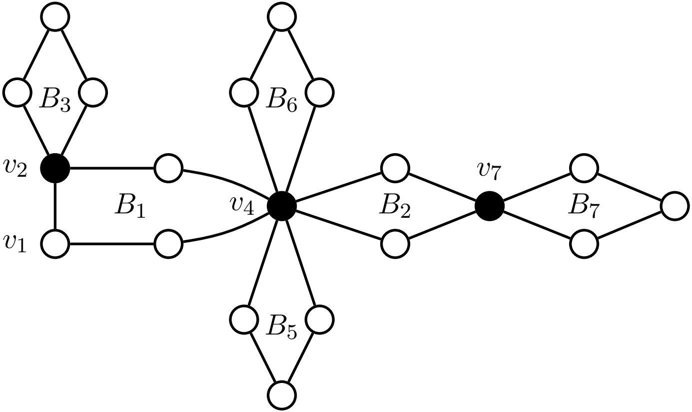
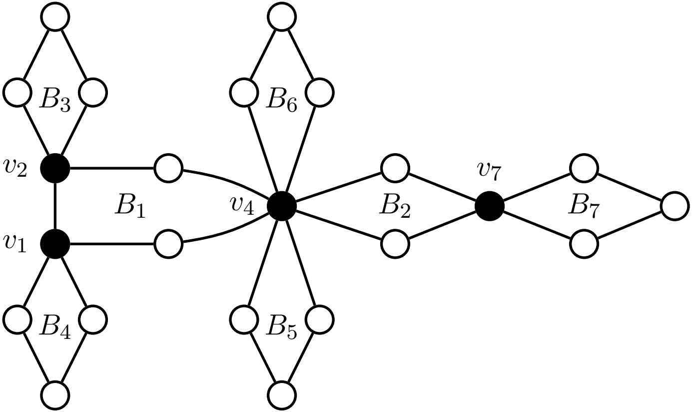
- For $v \in C(G)$ and $B \in \mathcal{B}(v)$
  - $G_{v,B}$ is the connected component of $G - (V(B) \setminus \{v\})$ containing v



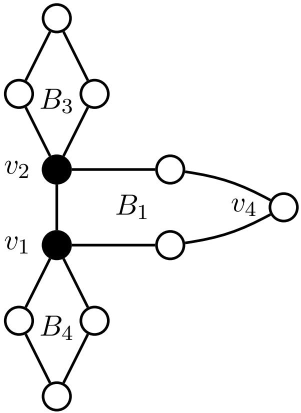
B_3
B_6
v_2
v_7
B_1
v_4
B_2
B_7
v_1
B_4
B_5

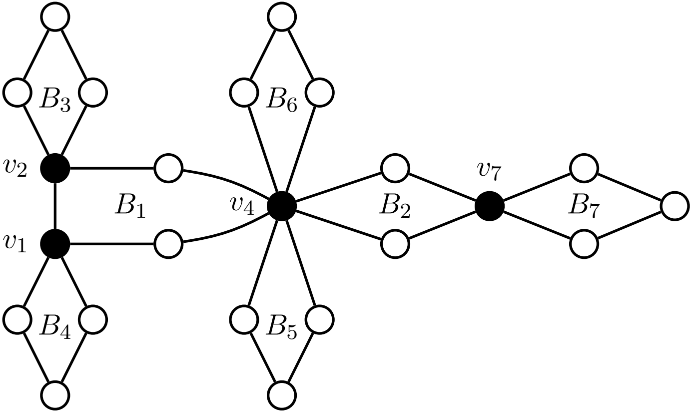
G_{B_1,v_1}



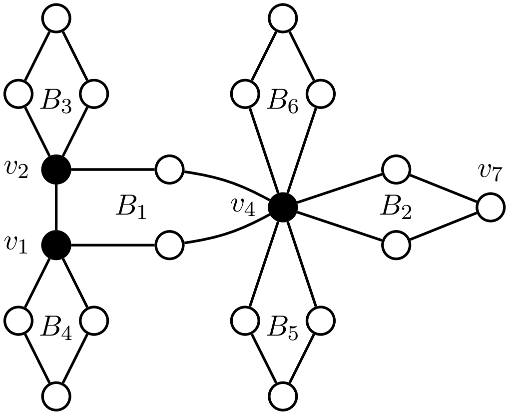

v_2
v_1
v_4
v_7
B_1
B_2
B_3
B_4
B_5
B_6
B_7

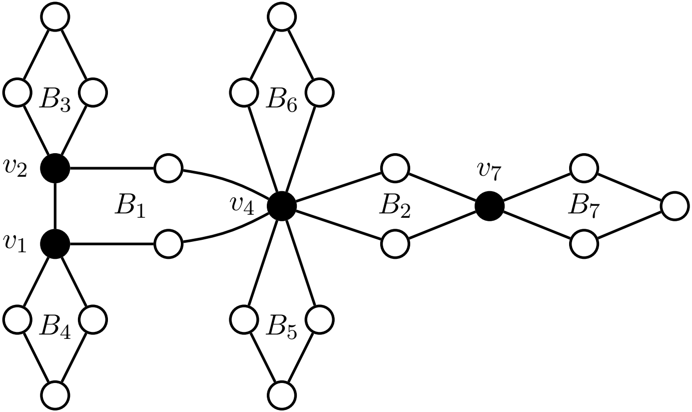
G_{B_1,v_4}



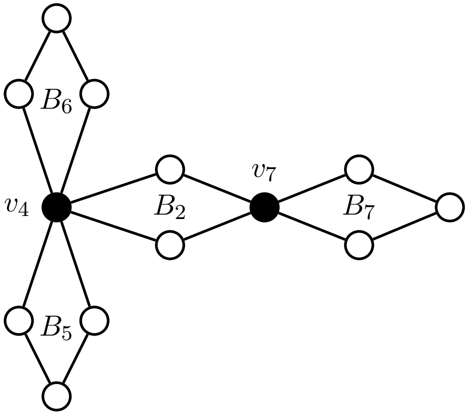

B_3
B_6
v_2
v_7
B_1
v_4
B_2
B_7
v_1
B_4
B_5

G_{v_7,B_7}




B_3
B_6
v_2
v_7
B_1
v_4
B_2
B_7
v_1
B_4
B_5

G_{v_4,B_1}



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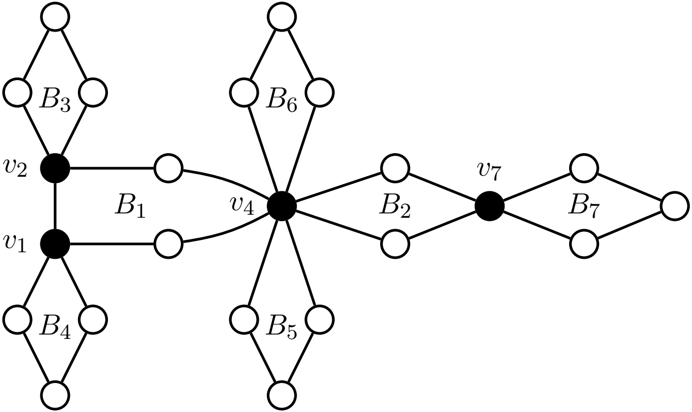
- $f'_{i,j} = |V(G_{B_i,v_j})|$.



B_3
B_6
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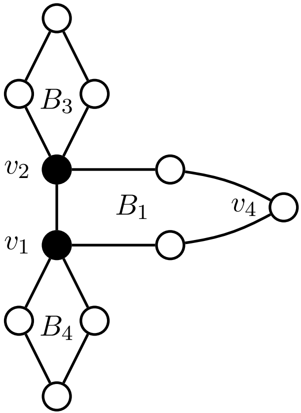
G_{B_1,v_4}



Convexity number of chordal graphs

Lemma

For any graph G, it holds $c_\Delta(G) = \max_{B_i \in \mathcal{B}(G)} \{f_i\}$.

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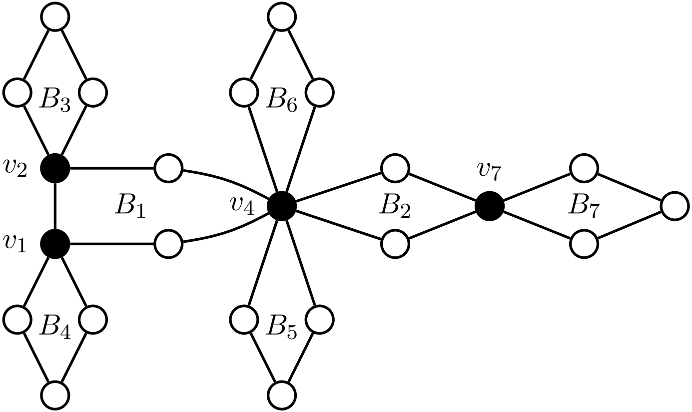
B_3
B_6
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B_2
B_7
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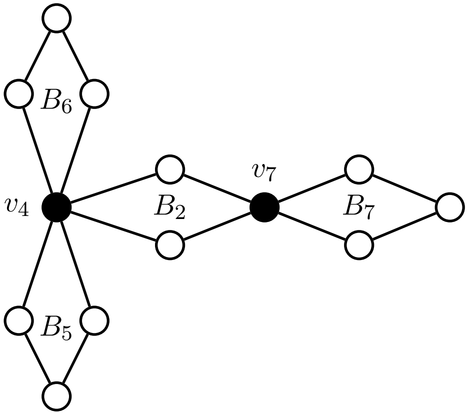
G_{v_4,B_1}



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(i) $g_{j,i} = \sum_{B_k \in \mathcal{B}(v_j) \setminus \{B_i\}} f_{k,j}$, *and*

(ii) $g'_{j,i} = 1 + \sum_{B_k \in \mathcal{B}(v_j) \setminus \{B_i\}} \left(f'_{k,j} - 1 \right)$.

Theorem (Anand, Anil, Changat, D., Ramla, 2020)

If G *is a* 2*-connected chordal graph, then every pair of adjacent vertices form a* Δ*-hull set of* G.

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If G is a 2-connected chordal graph, then every pair of adjacent vertices form a Δ-hull set of G.

Theorem (Frank, 1976)

A maximum weighted independent set can be found in linear time for a chordal graph.

Convexity number of chordal graphs

- B'_i is the graph obtained from a copy of B_i by adding a pendant vertex u_j to every $v_j \in C(B_i)$ and assigning weight $g'_{i,j}$ to v_j, weight $g_{i,j}$ to u_j, and weight 1 to the remaining vertices.

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- $B''_{i,j} = B'_i - \{u_j, v_j\}$

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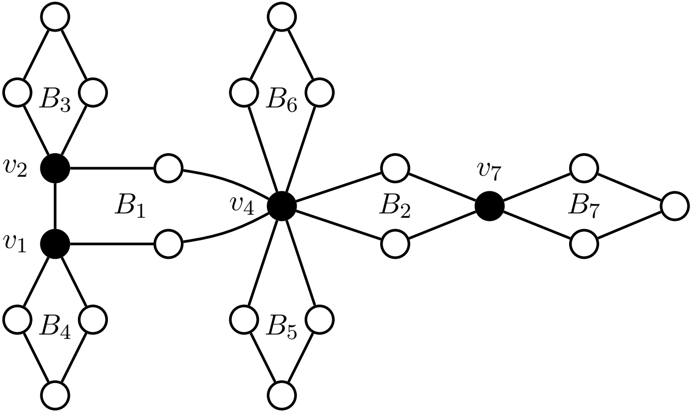
(iii) $$f'_{i,j} = |V(B_i)| + \sum_{v_k \in C(B_i) \setminus \{v_j\}} \left(g'_{k,i} - 1\right)$$



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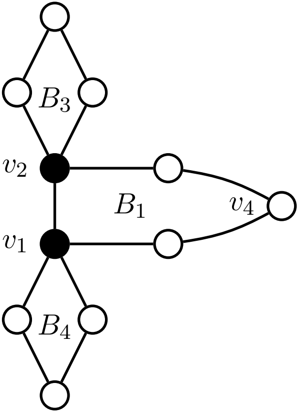

- Obrigado!