Minimizando comprimentos distintos em modelos de grafos de intervalo

Seminário Brasileiro de Grafos, Algoritmos e Combinatória

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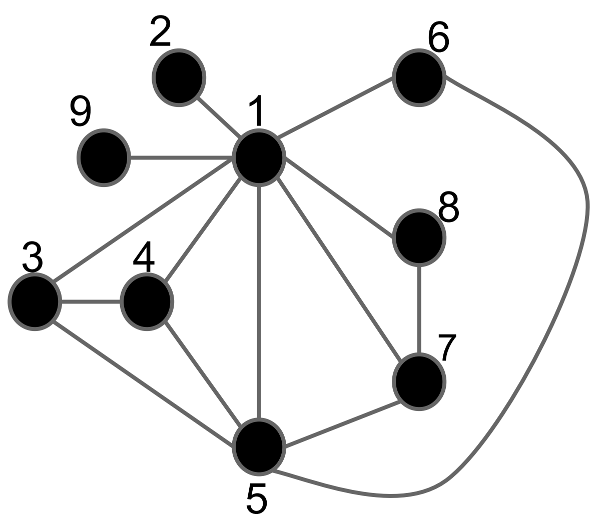
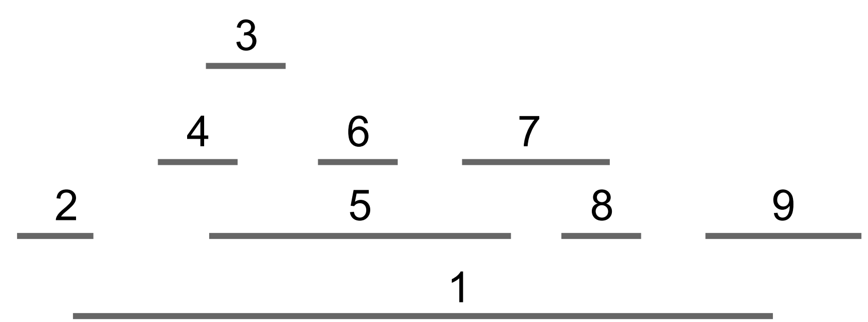
Interval Graphs

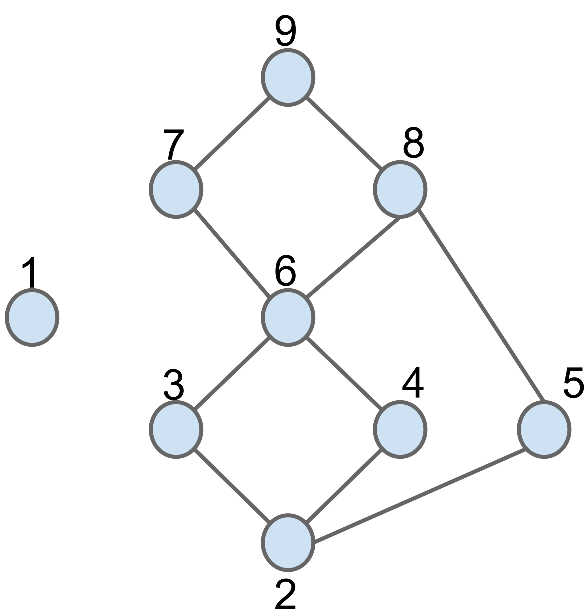
A graph G is *interval* if $\exists$ model $\{I_v : v \in V(G)\}$ such that, for all $u \neq v$, $uv \in E(G) \Leftrightarrow I_u \cap I_v \neq \varnothing$



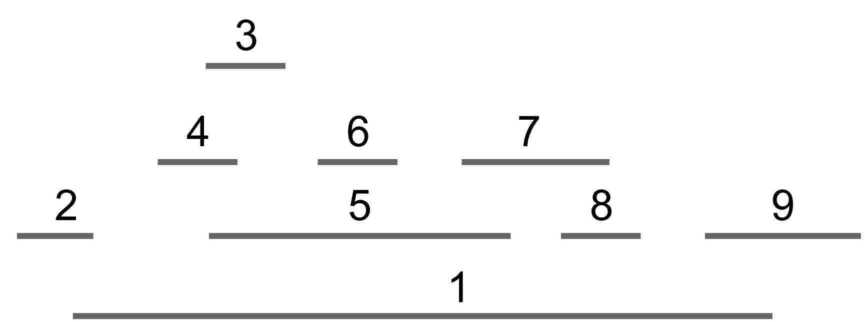


Interval Orders

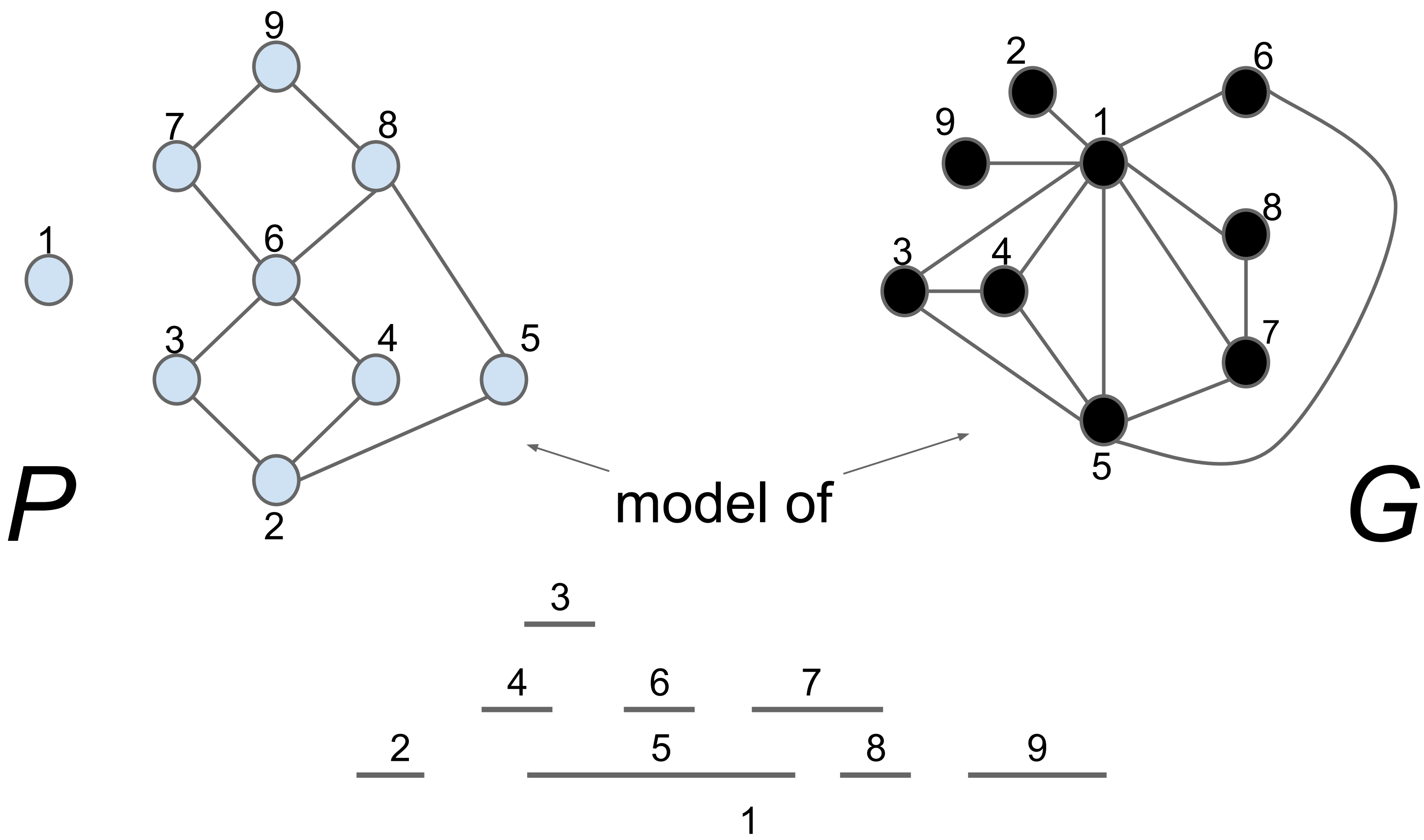
An order $P = (X, \prec)$ is *interval* if $\exists$ model $\{I_v : v \in X\}$
such that $u \prec v \Leftrightarrow I_u$ precedes I_v



(Hasse diagram)



P agrees with *G*

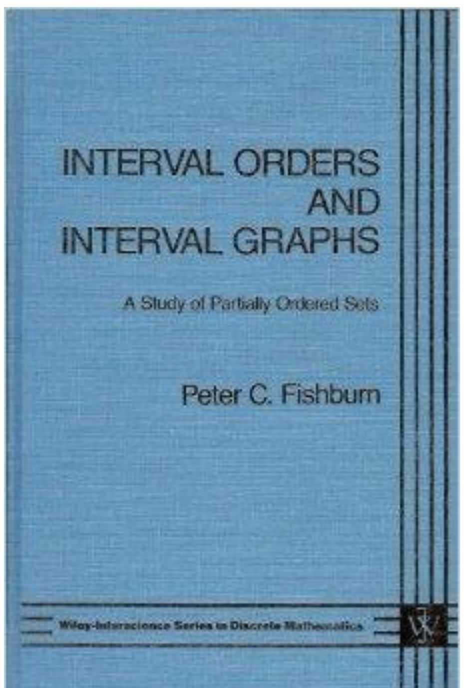


Applications

- planning [Allen ’93]
- task allocation [Papadimitriou ’79]
- archeology [Kendall ’69]
- temporal logic [Allen ’90]
- medical diagnosis [Klaus ’91]
- VLSI design [Ward ’90]
- genetics [Seymour, Benzer ’59]
  [Carrano ’88, Karp ’93]
- behavioral psychology [Coombs, Smith ’73]

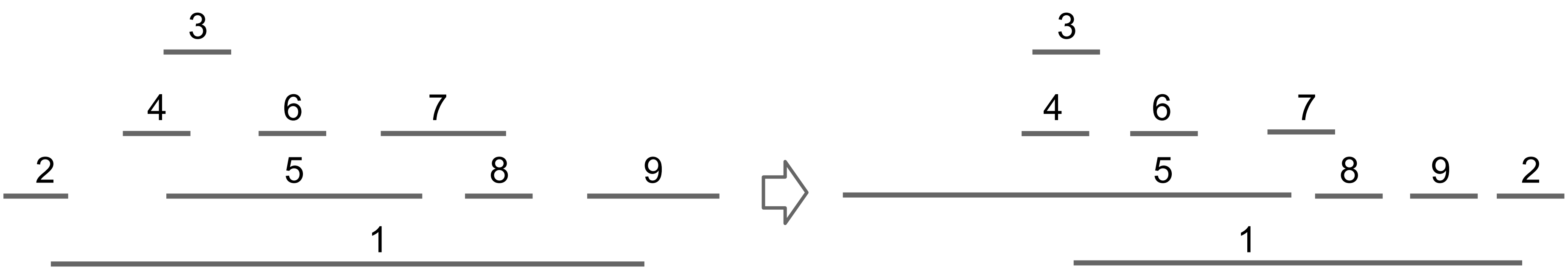
Well studied

- dozens of articles about interval orders and interval graphs
- a book only about them!



Interval Count

- Models of a given graph *G* may use many different interval sizes — but **how many is the minimum**? (= *interval count* IC(*G*))



4 sizes

2 sizes

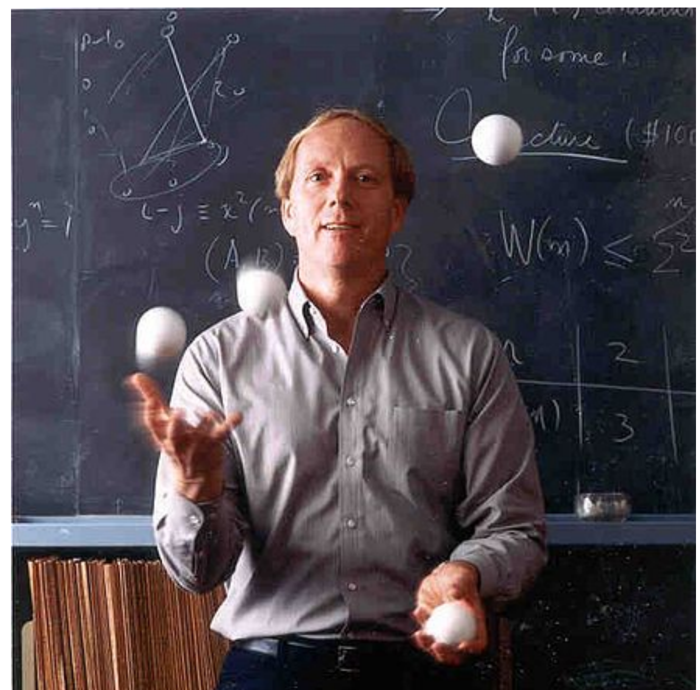
1 size??

Interval Count

- history
- interval count one
- interval count two+
- forbidden subgraphs
- extremal problems
- algorithmic upper bounds
- problems on graphs with limited interval count

Interval Count

- Problem suggested by R. Graham to R. Leibowitz during her PhD thesis



Interval Count

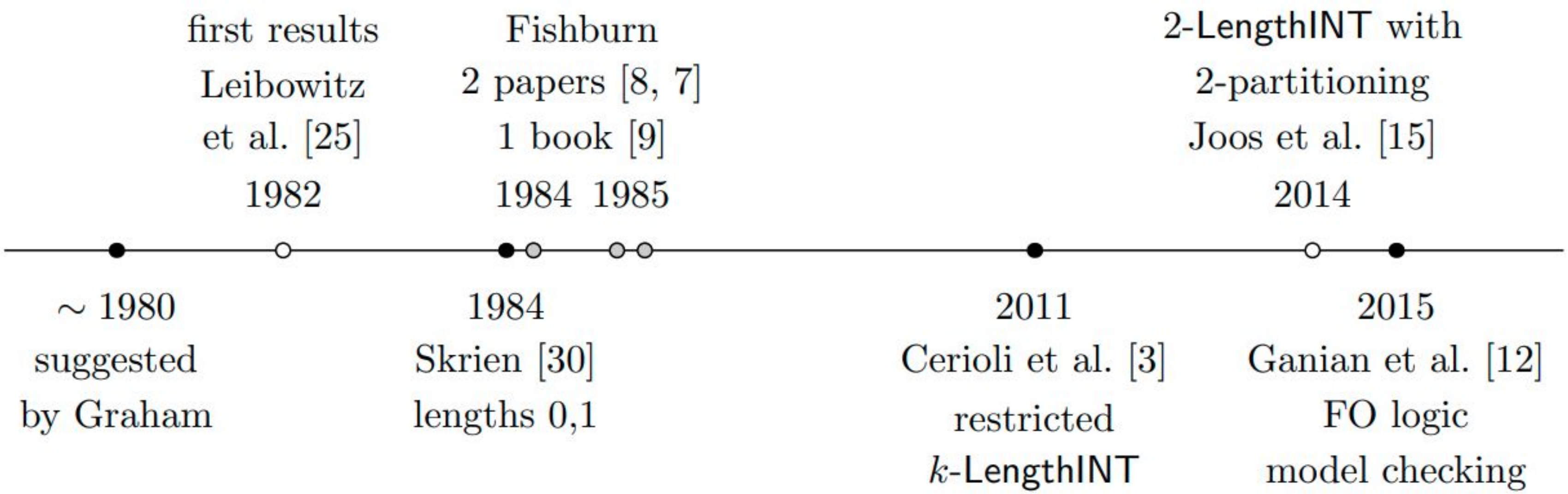


Fig. 3. Timeline of results for k-LengthINT. Notice a big gap between 1985 and 2011.

[Klavík, '17]

Interval Count

IC(R) is the *number of distinct sizes* in model R

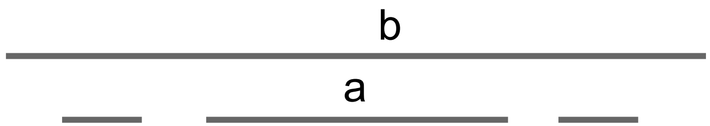
$\text{IC}(P) = \min \{ \text{IC}(R) : R \text{ is a model of } P \}$

$\text{IC}(G) = \min\{ \text{IC}(R) \mid R \text{ is a model of G} \}$

P agrees with $G \Rightarrow \text{IC}(G) \leq \text{IC}(P)$

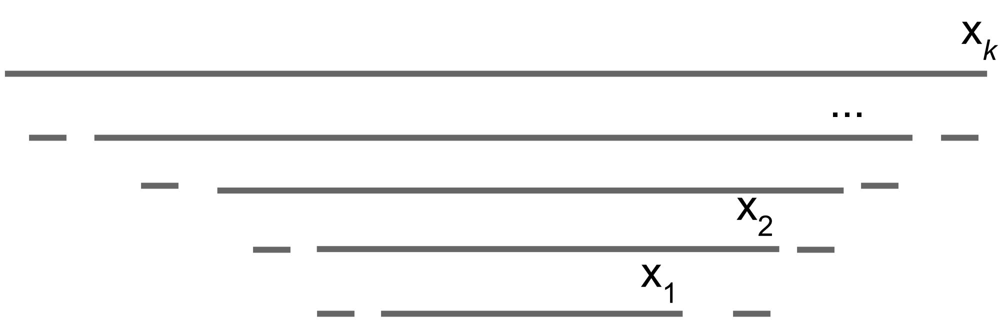
Interval Count

- *Nested intervals* must be assigned different sizes!



I_a is nested into I_b

- nesting depth of R = max { k : x_1, x_2, …, x_k is a chain of consecutively nested intervals in R }



$|I_{x1}| < |I_{x2}| < \ldots < |I_{xk}|$

Interval Count

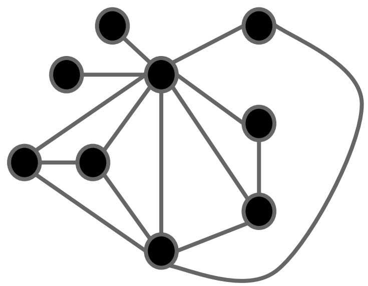
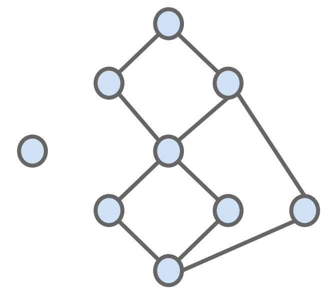
- *nesting depth of P* = nesting depth of a model *R* of *P*
- *nesting depth of G* =
  min { nesting depth of *R* : *R* is a model of *G* }

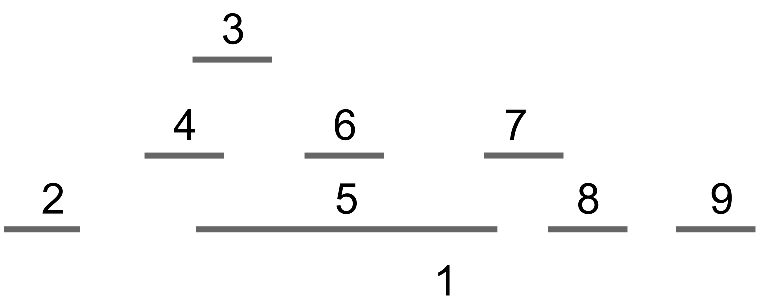
- *k-nested interval graphs = interval graphs whose nesting depth is at most k*

Important relation to interval count:

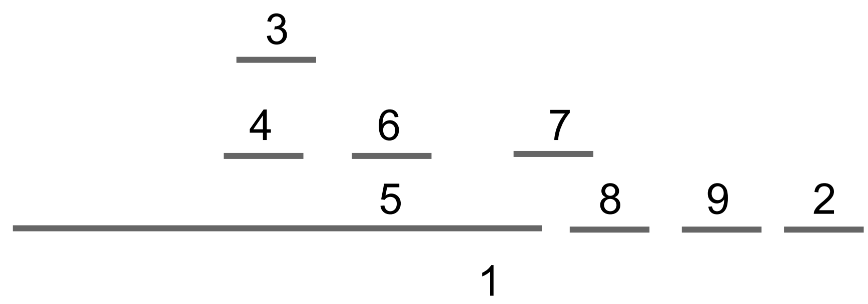
- IC(*P*) ≥ nesting depth of *P*
- IC(*G*) ≥ nesting depth of *G*

Interval Count

Let G =  and P = 



model of P and G



model of G (but not of P!)

IC(P) = 3 (why?)
IC(G) = 2 (why?)

k-Nested Interval Graphs

- The nesting depth of a graph *G* can be determined in polynomial time [Cerioli, Oliveira, Szwarcfiter '12] (in linear time by [Klavík, Otachi, Sejnoha, '19])

Interval Count One

- deciding whether IC(G) = 1 is a well solved problem
  - IC(G) = 1 ⇔ G is unit interval [Roberts '69] [Figueiredo, Meidanis, Mello '95] [Corneil '04]
  - IC(G) = 1 ⇔ G is $K_{1,3}$-free [Roberts '69]

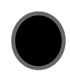
Interval Count Two+

- for all $k \geq 2$, it is **open** the complexity of deciding whether

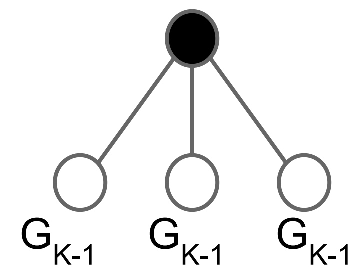
$$\mathrm{IC}(G) = k \quad \text{or} \quad \mathrm{IC}(P) = k$$

Interval Count

- IG(*G*) can be arbitrarily large



G_1



$G_{k,\, i \geq 2}$

$$IC(G_k) = k$$

Interval Count

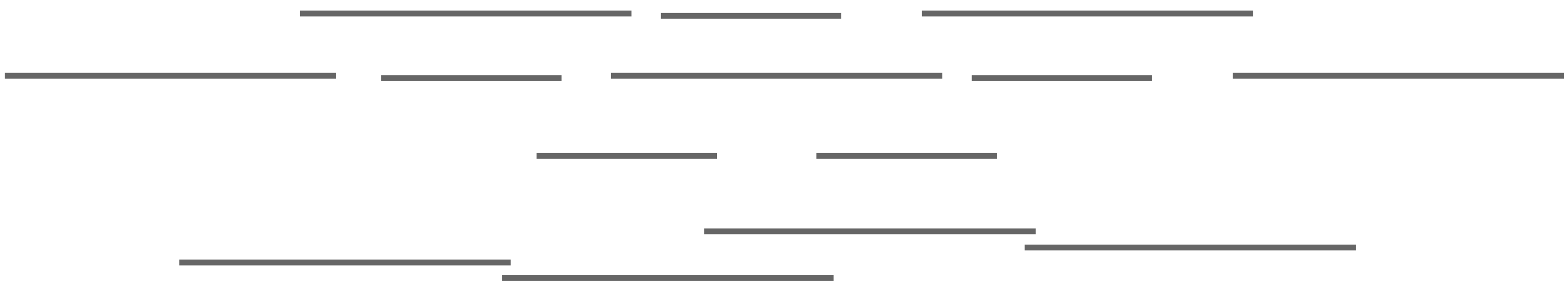
- IG(G) can be arbitrarily large even with constant nesting depth

Interval Count

$|\text{shorter}| > \frac{1}{3} |\text{longer}|$

Interval Count

$|\text{shorter}| > \frac{1}{3} |\text{longer}|$

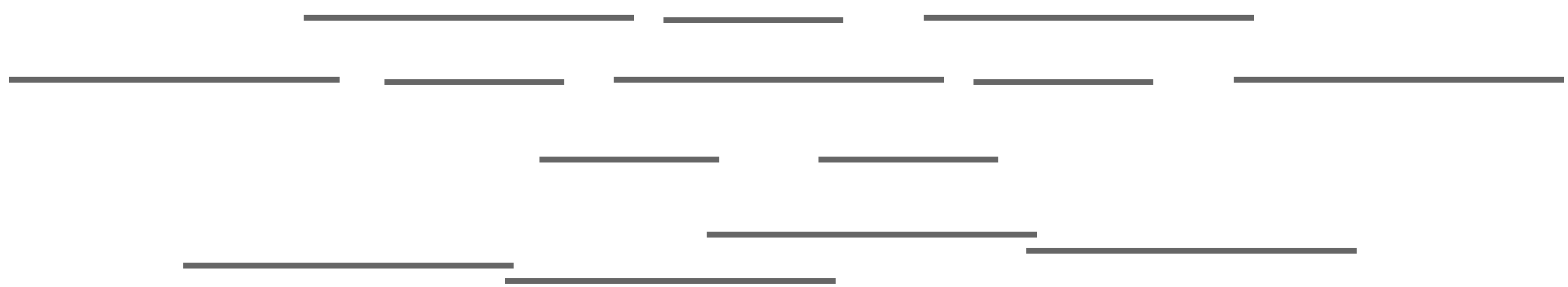


$|\text{shorter}| < \frac{1}{4} |\text{longer}|$

(a different component)

Interval Count

IC(G) = 3, nesting depth = 2



(a different component)

Interval Count

- There are examples of graphs having nesting depth 2 with arbitrarily large interval count [Fishburn '85]

Interval Count Two

- IC(G) ≤ 2 if G is a tree, threshold, or $K_{1,3}$-free except for one-vertex removal
  [Leibowitz ’82]

Interval Count Two

- A {*a*,*b*}-*model* has only intervals of sizes a or b
  LEN(a,b) is the class of graphs admitting {a,b}-models

- Recognition of LEN(0,1) in time $O(n^3)$ [Skrien '84] and $O(n + m)$ [Rautenbach, Szwarcfiter '11]

- Characterization of orders agreeing with graphs in LEN(1,2) by forbidden subgraphs assuming a preassignment of sizes to vertices
  [Boyadzhiyska, Isaak and Trenk '17]

Interval Count Two

- Of course, LEN(a,b) = LEN(ak,bk) for all $k > 0$…
- … but LEN(a',b') $\nsubseteq$ LEN(a,b) when $b'/a' \neq b/a$
  [Francis, Medeiros, Oliveira, Szwarcfiter (*to appear*)]

Interval Count Two

- Let:

$AS(G) = \{$ the longest size in R :
R is a model of G with $IC(R) = 2$ and
smaller size = 1 $\}$

Ex.: $AS(K_{1,r}) = (r\text{-}2, \infty)$

Characterization for LS(G) in general?

Interval Count Two

- W. Trotter conjectured LS(*G*) to be a real interval

- Fishburn disproved the conjecture, by showing a family of graphs for which LS(G) is a disjoint union of an arbitrary number of intervals [Fishburn '85]

Arbitrary Interval Count

- Graham initially conjectured that $\mathrm{IC}(G \setminus x) = k \Rightarrow \mathrm{IC}(G) \leq k+1$

- [Leibowitz '82] demonstrated that the conjecture holds for $k = 1$

Arbitrary Interval Count

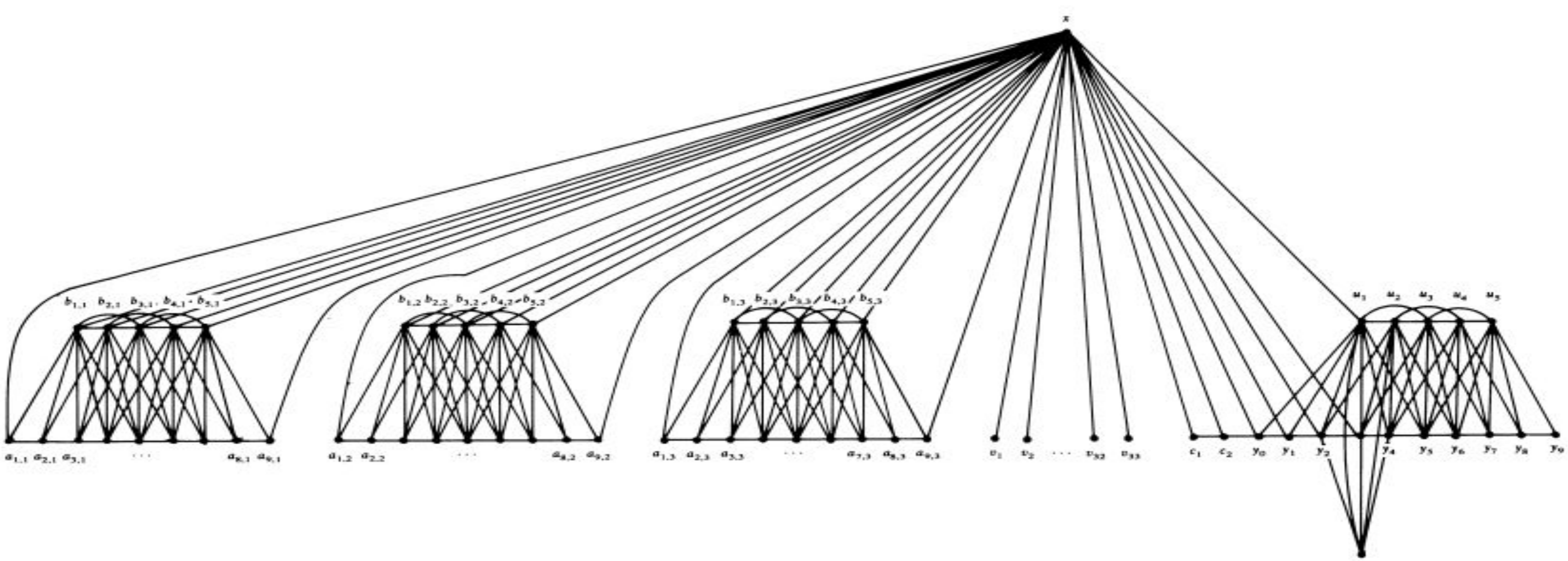
- But the conjecture fails in general: there is a graph G for which

$\mathrm{IC}(G) = \mathrm{IC}(G \setminus x) + 2$ [Leibowitz, Assmann, Peck ’82]

Arbitrary Interval Count

- But the conjecture fails in general: there is a graph *G* for which

IC(*G*) = IC(*G* \ *x*) + 2 [Leibowitz, Assmann, Peck ’82]



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G. W. Peck

From Wikipedia, the free encyclopedia

This article is about a pseudonymous attribution. For people named G. W. Peck, see G. W. Peck (disambiguation).

G. W. Peck is a pseudonymous attribution used as the author or co-author of a number of published mathematics academic papers. Peck is sometimes humorously identified with George Wilbur Peck, a former governor of the US state of Wisconsin.[1]

Peck first appeared as the official author of a 1979 paper entitled "Maximum antichains of rectangular arrays".[2] The name "G. W. Peck" is derived from the initials of the actual writers of this paper: Ronald Graham, Douglas West, George B. Purdy, Paul Erdős, Fan Chung, and Daniel Kleitman. The paper initially listed Peck's affiliation as Xanadu, but the editor of the journal objected, so Ron Graham gave him a job at Bell Labs. Since then, Peck's name has appeared on some sixteen publications,[3] primarily as a pseudonym of Daniel Kleitman.[1]

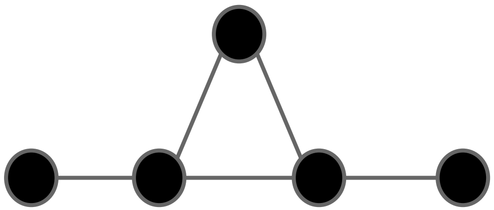
Arbitrary Interval Count

- If IC(G) = k $\Leftrightarrow$ G does not have any graph in L as an induced subgraph, then L is infinite, for $k \geq 2$ [Fishburn '85]

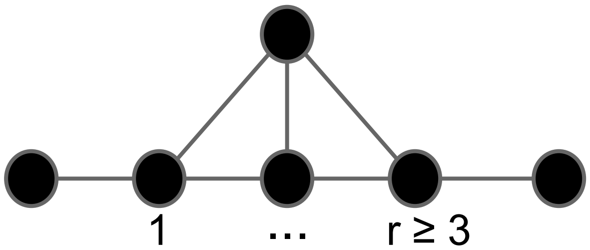
(Note the characterization of unit interval graphs by forbidden subgraphs)

Arbitrary Interval Count

- If the graph *G* (resp. order *P*) is extended-bull-free, then IC(*G*) (resp. IC(*P*)) equals to the nesting depth of *G* [Cerioli, Oliveira, Szwarcfiter '06]



bull



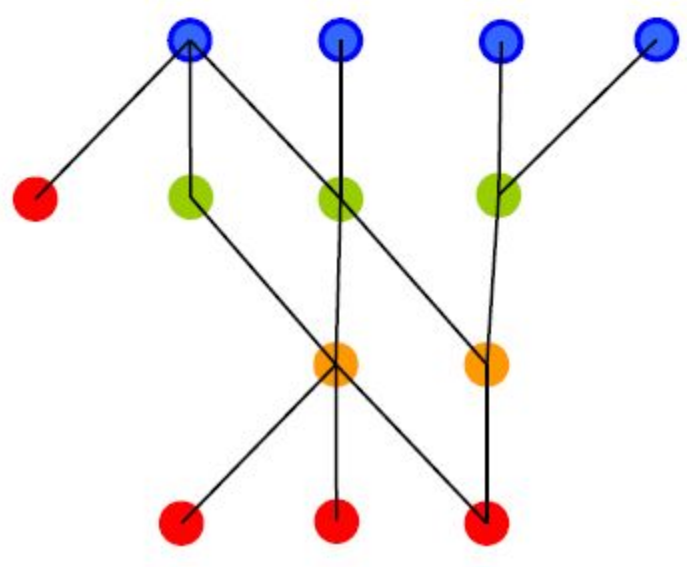
extended-bull

Arbitrary Interval Count

- **Lemma**: In a model *R* of an extended-bull-free graph, an interval *I* can have its size increased arbitrarily in *R*, increasing as a consequence only a subset of intervals in which *I* is nested

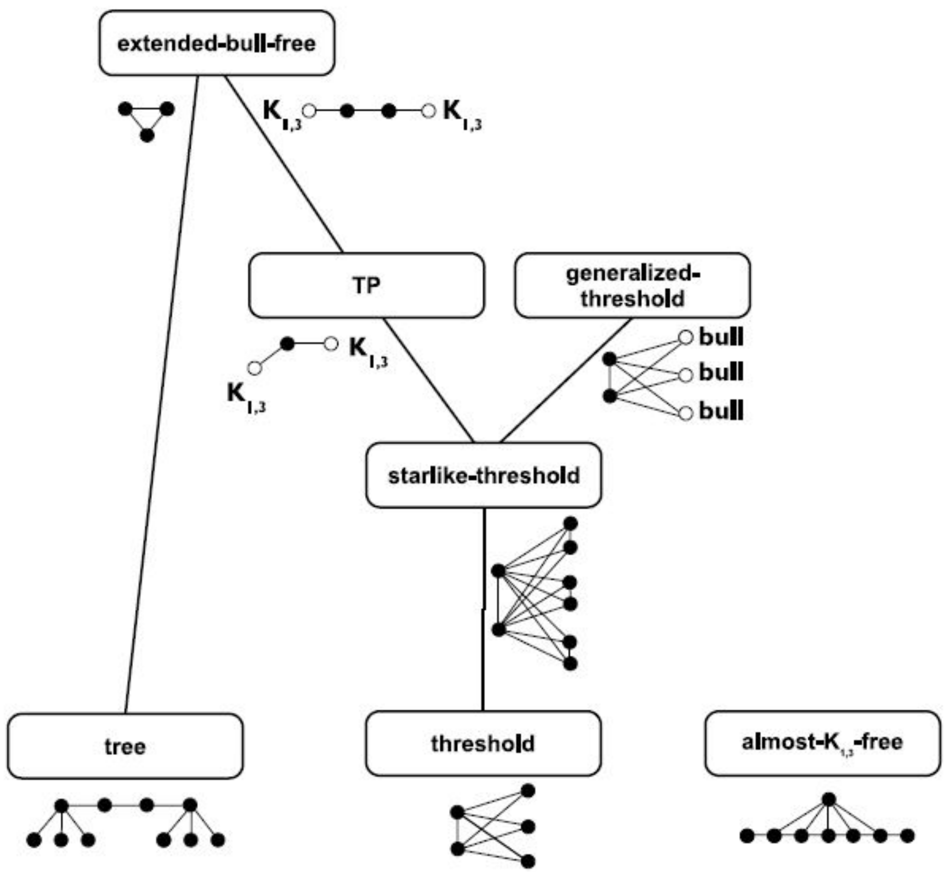
Arbitrary Interval Count

- Take any model *R* of *G*. Let the order (V(*G*), $\prec$) be such that $u \prec v \Leftrightarrow I_u$ is included in I_v
- For each height *h = 1,2,...*, take a smallest interval of height *h*. Increase it until its size becomes the size of the largest of that level. Take a next smallest interval until all intervals of height h have the same size



Arbitrary Interval Count

- IC restricted to classes (inclusion diagram)



Arbitrary Interval Count

SizeAssignment(G**,** $\{S_v : v \in V(G)\}$**)**

Input:

an interval graph G and, for each $v \in V(G)$, a real value S_v

Question:

is there an interval model $\{I_v : v \in V(G)\}$ of G such that $|I_v| = S_v$?

- **SizeAssignment** is NPC [Pe'er, Shamir '97]

Interval Count with Partition

IC2-Part(*G*, *A*, *B*):

Input:

a graph G and a bipartition $A \cup B = V(G)$

Question:

are there a model $\{I_x : x \in V(G)\}$ of G and real numbers $L_A < L_B$ such that $|I_a| = L_A$ for all $a \in A$ and $|I_b| = L_B$ for all $b \in B$?

Interval Count with Partition

Theorem (Joos, Löwenstein, Oliveira, Rautenbach, Szwarcfiter '14):
IC2-Part(G, A, B**)** is efficiently solvable when $G[A]$ and $G[B]$ are connected.

Interval Count with Partition

Lemma (Cerioli, Oliveira, Szwarcfiter '11):
If IC(G) = k, there is a model R of G with IC(R) = k such that all extreme points of intervals in R are distinct integer points.

Interval Count with Partition

- So, we can use Linear Programming (LP) for solving **IC2-Part(***P*, *A*, *B***)** for orders:

$$r_u + 1 \leq \ell_v \qquad \forall\ u \prec v\,,$$
$$r_u \geq \ell_v \qquad \forall\ u\ /\!/\ v\,,$$
$$r_a - \ell_a = L_A \qquad \forall\ a \in A\,,$$
$$r_b - \ell_b = L_B \qquad \forall\ b \in B\,,$$
$$L_A \geq 1,\ \ L_B \geq L_A + 1,$$
$$\ell_u \geq 1,\ \ r_u \geq 1$$

Interval Count with Partition

- The remaining of the proof is to completely characterize the precedence relation that can be assumed

Extremal Problems

- **[Fishburn '85]** proposed the functions:

  $\sigma(k) = \min \{ |P| : P \text{ is an order such that } IC(P) \geq k \}$
  $\tau(k) = \min \{ |V(G)| : G \text{ is a graph such that } IC(G) \geq k \}$
  (clearly, $\sigma(k) \leq \tau(k)$)

Extremal Problems

- **[Fishburn '85]** proposed the functions:

  $\sigma(k) = \min \{ |P| : P \text{ is an order such that } IC(P) \geq k \}$
  $\tau(k) = \min \{ |V(G)| : G \text{ is a graph such that } IC(G) \geq k \}$
  (clearly, $\sigma(k) \leq \tau(k)$)

  and proved that

  $2k \leq \sigma(k) \leq 3k - 2$ for all $k \geq 2$
  $\therefore IC(P) \leq \lfloor n/2 \rfloor$ (n = |P|; q = number of cliques)
  $IC(G) \leq \lfloor (q+1)/2 \rfloor$ **[Cerioli, Oliveira, Szwarcfiter '11]**

Extremal Problems

- **[Fishburn '85]** conjectured:

  $\sigma(k) = 3k - 2$ for all $k \geq 2$

  and prove the conjecture for all $k \leq 7$

- **[Francis, Medeiros, Oliveira, Szwarcfiter (to appear)]** proposed:

$\sigma_C(k) = \min \{ |P| : P$ is an order of class C such that $IC(P) \geq k \}$
$\tau_C(k) = \min \{ |V(G)| : G$ is a graph of class C such that $IC(G) \geq k \}$

Extremal Problems

Function \ C	**TP**	**Split**
$\sigma_C(k)$	$3k - 2$	$3k - 2$
$\tau_C(k)$	$7 \cdot 2^{k-1} - 3$	$3k - 1$

[Francis, Medeiros, Oliveira, Szwarcfiter (to appear)]

Algorithmic upper bound

- quadratic programming has been used to derive an algorithmic upper bound on the interval count of orders

- empirically very close to the actual values

[Medeiros, Oliveira, Szwarcfiter '21]

Algorithmic upper bound

- Given an order P:

$$
\begin{aligned}
\textbf{min} \quad & \textstyle\sum_{u=1,\dots,n} (s_u - 1)^2 \\
\textit{s.t.} \quad & r_u + 1 \le \ell_v && \forall\ u \prec v \\
& r_u \ge \ell_v && \forall\ u \parallel v \\
& r_u - \ell_u = s_u && \forall\ u \\
& \ell_u \ge 0,\ r_u \ge 0,\ s_u \ge 1
\end{aligned}
$$

Algorithmic upper bound

Tabela 1: Resultados parciais sumarizados.

			Limite Superior Algorítmico		Tempo de Execução (s)	
Classe de P	n	$IC(P)$	$f_1(x)$	$f_2(x)$	$s_{f_1(x)}$	$s_{f_2(x)}$
1	13	2	3	4	0,031	0,047
2	9	2	2	2	0,031	0,062
3 (TP)	7	3	3	3	0,015	0,031
3 (TP)	43	15	15	15	0,150	0,310
3 (TP)	40	4	4	4	0,093	0,109
3 (TP)	121	5	5	6	0,921	1,249
3 (SPLIT)	14	5	5	6	0,015	0,031
3 (SPLIT)	74	25	25	26	0,340	1,658
3 (SPLIT)	16	6	6	7	0,031	0,032
3 (SPLIT)	28	10	10	10	0,059	0,046
4 (CL)	12	4	5	4	0,037	0,066
4 (CL)	18	6	6	6	0,037	0,045
4 (CL)	30	10	10	10	0,078	0,117
5	25	≥ 3	7	8	0,104	0,060

Problems on Limited IC Graphs

- There are several algorithms for problems on interval graphs and/or unit interval graphs

Problems on Limited IC Graphs

Interval

Problems

Problems in italics have no summary page and are only listed when ISGCI contains
class.

Parameter decomposition

book thickness decomposition [?]	Unknown to ISGCI
booleanwidth decomposition [?]	Polynomial
cliquewidth decomposition [?]	Unknown to ISGCI
cutwidth decomposition [?]	Unknown to ISGCI
treewidth decomposition [?]	Polynomial

Unweighted problems

3-Colourability [?]	Linear
Clique [?]	Polynomial
Clique cover [?]	Linear
Colourability [?]	Linear
Domination [?]	Linear
Feedback vertex set [?]	Linear
Graph isomorphism [?]	Linear
Hamiltonian cycle [?]	Linear
Hamiltonian path [?]	Polynomial
Independent dominating set [?]	Linear
Independent set [?]	Linear
Maximum cut [?]	NP-complete
Monopolarity [?]	Linear
Polarity [?]	Polynomial
Recognition [?]	Linear

Weighted problems

Weighted clique [?]	Polynomial
Weighted feedback vertex set [?]	Linear
Weighted independent dominating set [?]	Polynomial
Weighted independent set [?]	Linear
Weighted maximum cut [?]	NP-complete

Unit Interval

Problems

Problems in italics have no summary page and are only listed when ISGCI
class.

Parameter decomposition

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booleanwidth decomposition [?]	Polynomial
cliquewidth decomposition [?]	Unknown to ISGCI
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Unweighted problems

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Feedback vertex set [?]	Linear
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Hamiltonian cycle [?]	Linear
Hamiltonian path [?]	Linear
Independent dominating set [?]	Linear
Independent set [?]	Linear
Maximum cut [?]	Unknown to ISGCI
Monopolarity [?]	Linear
Polarity [?]	Polynomial
Recognition [?]	Linear

Weighted problems

Weighted clique [?]	Linear
Weighted feedback vertex set [?]	Linear
Weighted independent dominating set [?]	Polynomial
Weighted independent set [?]	Linear
Weighted maximum cut [?]	NP-complete

Problems on Limited IC Graphs

Theorem [Adhikary et al, '21]:
MaxCut on interval graphs is NPC.

Theorem [de Figueiredo, Melo, Oliveira, Silva, '21]:
MaxCut on interval graphs G with $\mathrm{IC}(G) = 4$ is NPC.

Open questions

Open Problems

- Leibowitz disproved the Graham conjecture by showing a graph whose IC decreases two units when one vertice is removed

  How about the weaker version of Graham conjecture: is it true that

  $\mathrm{IC}(G \setminus x) = k \Rightarrow \mathrm{IC}(G) \leq k + r$,
  for some constant $r \geq 2$?

Open Problems

- Recognition of the interval count on more general interval graph subclasses (more general than extended bull free graphs)

Open Problems

- Characterization of the class of graphs *G* for which the IC(*G*) equals its nesting depth

- Regarding **IC2-Part**, can we get rid of the LP and use a more theoretic graph approach?

Open Problems

- **MaxCut** on interval graphs G having IC(G) ≤ 3?

- Is the empirical upper bound actually equal to the interval count on particular subclasses?

Obrigado
pela atenção!