Jogo de Coloração em Grafos e suas Variantes

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Introduction

Proper coloring

- The vertices of graph are colored
- Two adjacent vertices must receive distinct colors
- $\chi(G)$: chromatic number (min number in a proper coloring)

Greedy coloring

- Proper vertex coloring / colors are integers
- Take an ordering of the vertices.
- A vertex must receive the minimum available color.
- $\Gamma(G)$: Grundy number (max number in a greedy coloring)

$$\chi(G) \leq \Gamma(G)$$

Two graph coloring games

- **Instance:** a graph G and a set C of colors/integers
- Two players **Alice** and **Bob** alternate their turns in choosing an uncolored vertex to be **proper colored** by an integer of C
- Alice **starts** and **she wins** if all vertices are successfully colored; Otherwise, Bob wins the game
- Zermelo-von Neumann Th.: Alice or Bob has a winning strat

Graph coloring game ($\chi_g(G) \geq \chi(G)$)

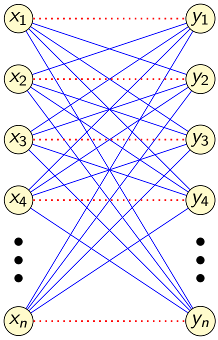
- Alice and Bob may use **any** possible integer of C
- **Game chromatic** number $\chi_g(G)$: minimum number of colors s.t. Alice has a winning strategy in the graph coloring game

Greedy coloring game ($\chi(G) \leq \Gamma_g(G) \leq \Gamma(G)$)

- Alice and Bob must use **the smallest** possible integer of C
- **Game Grundy number** $\Gamma_g(G)$: minimum number of colors s.t. Alice has a winning strategy in the greedy coloring game

An example for both coloring games

- Complete bipartite graph without a matching
- **If Alice is the first** to play, Bob can force n colors: just play in the non-neighbor of Alice's last vertex.
- **If Bob is the first** to play, Alice wins with 2 colors.
  Normal game: Alice colors non-neighbor with other color.
  Greedy game: color same side of Bob's first vertex.



Known results
Graph coloring game / Game chromatic number $\chi_g(G)$

- First considered by [Brams] and described by [Gardner'81, Math. Games column of Scientific American]
- Reinvented by [Bodlaender'91]: "*The complexity of the Color Construction Game is an interesting open problem*"
- forest ≤ 4 [Faigle...'93], outerplanar ≤ 7 [Kierstead...'94]
- $\chi_g \leq (\chi_a + 1)^2$ acyclic chromatic number χ_a [Dinski,Zhu'99]
- $\chi_g(P_k) \leq 3k + 2$ for partial k trees [Zhu'00]
- $\chi_g(G) \leq 5$ in cacti [Sidorowicz'07]
- Asympt. behavior $\chi_g(G(n, p))$ [Bohman, Frieze, Sudakov'08]
- Ex.value χ_g cartes prod K_2 w path/cycle/clique [Bartnick'08]
- Planar graphs: $\chi_g \leq 17$ [Zhu'08], $\chi_g \leq 13$ [Sekiguchi'14, girth$\geq$ 4], $\chi_g \leq 5$ [Nakprasit'18, girth$\geq$ 7]

- $\chi_g(F)$ poly forests no vertex deg 3 [Dunn et. al'15]: "*more than two decades later,* ***this question*** *remains open*".
- poly characterization game-perfect graphs [Andres,Lock'19]: "*the question of PSPACE-hardness remains open*".

Known results

Greedy coloring game / Game Grundy number $\Gamma_g(G)$

- Introduced by [Havet, Zhu'13]
- $\Gamma_g(G) = \chi(G)$ in cographs [Havet,Zhu'13]
- $\Gamma_g(F) \leq 3$ in forests [Havet, Zhu'13]
- $\chi_g(G) \leq 7$ in partial 2-trees [Havet, Zhu'13]
- Two questions of [Havet,Zhu'13]
  - (*) $\chi_g(G)$ is upper bounded by a function of $\Gamma_g(G)$?
  - (**) $\Gamma_g(G) \leq \chi_g(G)$ for every graph G?
- (*) = NO [Krawczyk,Walczak'15]
- (**) is still open

Our results-1

Complexity results

- $\chi_g(G)$ is PSPACE-hard answer Bodlaender'91 open question
- $\Gamma_g(G)$ is PSPACE-hard
- Both decision problems are PSPACE-Complete

Exact/algorithmic results

- $\Gamma_g(G) = \chi(G)$ poly for split graphs
- $\Gamma_g(G) = \chi(G)$ poly for extended P_4-laden graphs, a class in the top of a hierarchy of graphs with few P_4's
- In both cases, Alice wins with $\chi(G)$ colors even if Bob can start the game and pass any turn

$\chi_g(G)$ is PSPACE-hard

Zhu'99 open question: Graph coloring game "*exhibits some strange properties*". Does Alice have a winning strategy with $k+1$ colors if she has one with k colors?

We define three decision problems for the graph coloring game:

- (Problem 1) Given G and k: $\chi_g(G) \leq k$?
- (Problem 2) Given G and k: Does Alice have a winning strategy with k colors?
- (Problem 3) Given G and $\chi(G)$: $\chi_g(G) = \chi(G)$?

Problems 1 and 2 are equivalent **iff** Zhu's question is true.
Problems 1 and 2 generalizations of Problem 3 - take $k = \chi(G)$

Problem 3 PSPACE-hard $\rightarrow$ Problems 1 and 2 PSPACE-hard

Reduce POSCNF $\rightarrow$ Problem 3: build G s.t. we know $\chi(G)$.

χ_g: Reduction from POS-CNF

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CNF formula, only positive variables, Alice and Bob alternate turns setting variables true or false. Alice wins if the formula is true.

Example

$(X_1 \vee X_2) \quad \wedge \quad (X_1 \vee X_3) \quad \wedge \quad (X_2 \vee X_4) \quad \wedge \quad (X_3 \vee X_4).$

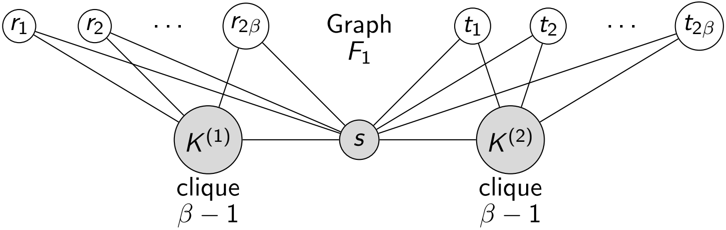
Bob has a winning strategy:

- X_1 True $\rightarrow$ X_4 False;
- X_2 True $\rightarrow$ X_3 False;
- X_4 True $\rightarrow$ X_1 False;
- X_3 True $\rightarrow$ X_2 False.

Good points

- POS-CNF is PSPACE-Complete
- If she/he has a winning strategy in POS-CNF, she/he also has a winning strategy if the opponent can pass turns.

χ_g: Important ingredient of the Reduction



Lemma:

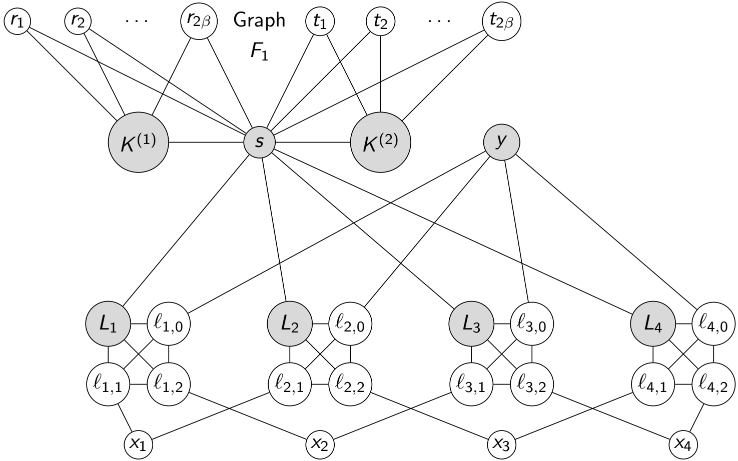
Alice has a winning strategy in F_1 with $2\beta - 1$ colors **iff** she colors vertex s first.

Proof:

If Alice does not color s first, Bob can color β vertices r_k/t_k, forcing 2β colors with clique $K^{(i)} \cup s$. If Alice colors s first, she can color $K^{(1)}$ and $K^{(2)}$ before Bob colors β vertices r_k/t_k.

χ_g: Reduction from POS-CNF

$$(X_1 \vee X_2) \quad \wedge \quad (X_1 \vee X_3) \quad \wedge \quad (X_2 \vee X_4) \quad \wedge \quad (X_3 \vee X_4).$$



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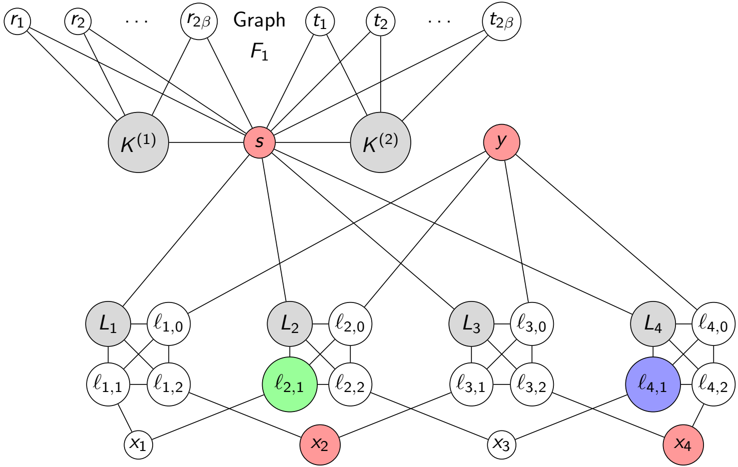


$\Gamma_g(G)$ is PSPACE-hard

Differently than the Graph Coloring Game, if Alice has a winning strategy with $k+1$ colors in the greedy coloring game she has one with k colors.

We define two decision problems for the greedy coloring game:

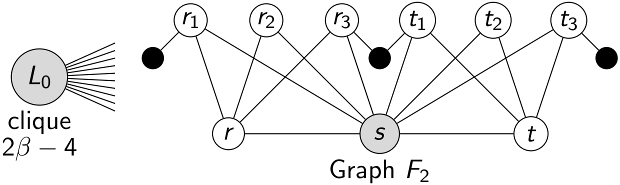
- (Problem 1') Given G and k: $\Gamma_g(G) \leq k$? That is: Does Alice have a winning strategy with k colors?
- (Problem 2') Given G and $\chi(G)$: $\Gamma_g(G) = \chi(G)$?

Problems 1' generalization of Problem 2' - take $k = \chi(G)$

Problem 2' PSPACE-hard $\rightarrow$ Problem 1' PSPACE-hard

Reduce POSCNF $\rightarrow$ Problem 2': build G s.t. we know $\chi(G)$.

Γ_g: Important ingredient of the Reduction



Lemma:

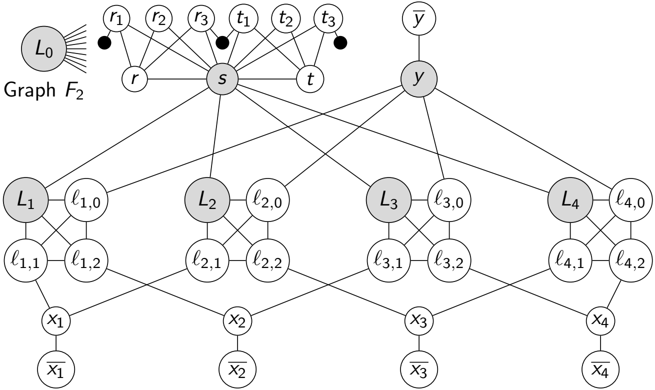
Alice has a winning strategy in F_2 with $2\beta - 1$ colors **iff** she colors vertex s first.

Proof:

If Alice does not color s first (assume r wlg), Bob colors t_2 and a black vertex with 1, forcing 4 colors in a triangle $s - t - t_i$. If Alice colors s first, she can color r and t with 2 or 3 (black vertices will be 1), forcing 3 colors.

Γ_g: Reduction from POS-CNF

$$(X_1 \vee X_2) \quad \wedge \quad (X_1 \vee X_3) \quad \wedge \quad (X_2 \vee X_4) \quad \wedge \quad (X_3 \vee X_4).$$



Positive results

[Havet,Zhu'13]: $\Gamma_g(G) = \chi(G)$ for cographs (no induced P_4)

Cographs: $\chi(G) = \Gamma(G)$. Then $\Gamma_g(G) = \chi(G)$ (even if Bob starts and can pass any turn).

Superclasses of cographs:

- P_4-sparse, P_4-laden, P_4-tidy
- $\Gamma(G)$ can be larger than $\chi(G)$ as much as desired

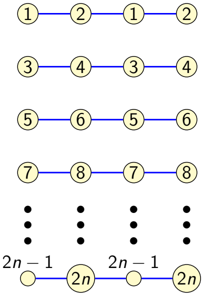
Split graphs:

- partition $V = C \cup S$: Clique C and indepedent set S
- $\chi(G) \leq \Gamma(G) \leq \chi(G) + 1$

We prove $\Gamma_g(G) = \chi(G)$ (even if Bob starts / can pass any turn) in split graphs and extended P_4-laden graphs.

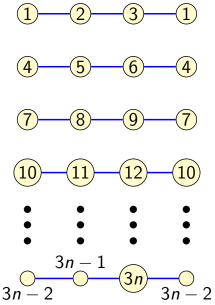
P_4-sparse example: join of n P_4's

P_4-sparse graph G



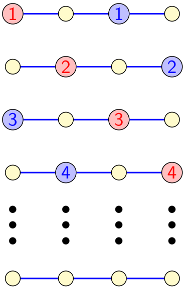
$\chi(G) = 2n$

P_4-sparse graph G



$\Gamma(G) = 3n$

P_4-sparse graph G



$\Gamma_g(G) = 2n$

Graph classes with few P_4's

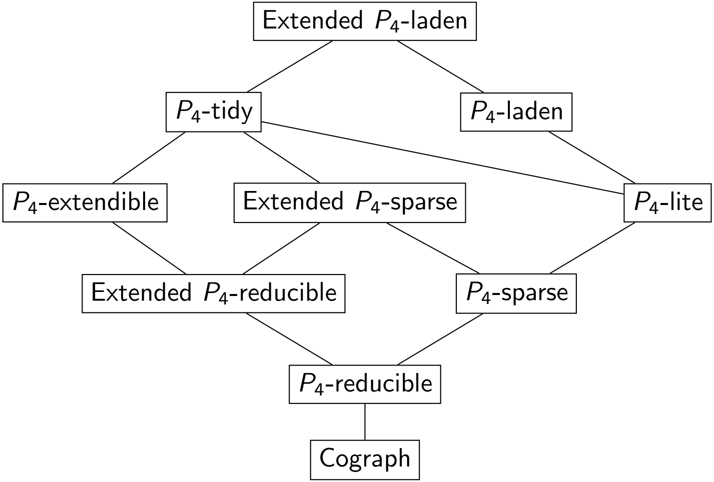


Figure: Hierarchy of graphs with few P_4's.

Extended P_4-laden graphs

Decomposition theorem [Giakoumakis'96]

G is extended P_4-laden **iff** one of the following holds:

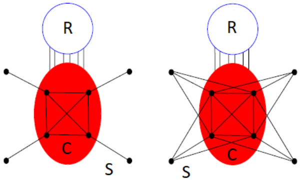
(a) G is the disjoint union or the join of two non-empty extended P_4-laden graphs;

(b) G is a quasi-spider or a pseudo-split graph (R, C, S) such that $G[R]$ is an extended P_4-laden graph;

(c) G is isomorphic to C_5, P_5, $\overline{P_5}$, or has at most one vertex.

Operations: union, join, spider

- **Union** $G = G_1 \cup G_2$: No edge between G_1 and G_2.
- **Join** $G = G_1 \vee G_2$: All edges between G_1 and G_2.



G is a **pseudo-split** (R, C, S) if:

- C induces a clique and S induces an independent set
- All edges from R to C and no edges from R to S

G is a **spider** if it is a pseudo-split (R, C, S) st:

- $C = \{c_1, \ldots, c_k\}$ and $S = \{s_1, \ldots, s_k\}$ for some $k \geq 2$
- **Thin spider**: s_i is adjacent to c_j if and only if $i = j$
- **Thick spider**: s_i is adjacent to c_j if and only if $i \neq j$

$\Gamma'_g(G)$: Bob can start and pass any turn

Let $\Gamma'_g(G)$ be the minimum number of colors st Alice has a winning strategy in the greedy coloring game even if Bob can start and pass any turn. $\chi(G) \leq \Gamma_g(G) \leq \Gamma'_g(G) \leq \Gamma(G)$.

Union and Join

- $\Gamma'_g(G_1 \cup G_2) \leq \max\{\Gamma'_g(G_1), \Gamma'_g(G_2)\}$;
- $\Gamma'_g(G_1 \vee G_2) \leq \Gamma'_g(G_1) + \Gamma'_g(G_2)$.

Pseudo-split, quasi-spider, C_5, P_5, $\overline{P_5}$

- G pseudo-split $(R, C, S) \implies \Gamma'_g(G) \leq \Gamma'(G[R]) + |C|$
- G quasi-spider or $G \in \{C_5, P_5, \overline{P_5}\} \implies \Gamma'_g(G) = \chi(G)$

Extended P_4-laden

Applying the decomposition, we prove by induction that $\Gamma'_g(G) = \chi(G)$ for any extended P_4-laden G.

Conclusion-1

Complexity results

- Game chromatic number $\chi_g(G)$ is PSPACE-hard, answering Bodlaender's 1991 open question
- Game Grundy number $\Gamma_g(G)$ is PSPACE-hard
- The Graph Coloring Game and the Greedy Coloring Game are PSPACE-Complete

Exact/algorithmic results for Γ_g

- $\Gamma_g(G) = \chi(G)$ poly for split graphs
- $\Gamma_g(G) = \chi(G)$ poly for extended P_4-laden graphs, a class in the top of a hierarchy of graphs with few P_4's
- In both cases, Alice wins with $\chi(G)$ colors even if Bob can start the game and pass any turn

THANK YOU !!

Variants of these coloring games

Graph YZ-coloring game $g_{YZ}(\chi_g^{YZ}(G) \geq \chi(G))$

- $Y \in \{A, B\}$ and $Z \in \{A, B, no\ one\}$
- Y starts the game and Z may pass turns
- Alice and Bob may use **any** possible integer of C
- YZ-**game chromatic** number $\chi_g^{YZ}(G)$: min number of colors s.t. Alice has a winning strategy in the YZ-coloring game
- We omit Z when it is "no one": $\chi_g^A(G) = \chi_g(G)$ is the original game chromatic number.

Greedy YZ-coloring game g^*_{YZ} $(\chi(G) \leq \Gamma_g^{YZ}(G) \leq \Gamma(G))$

- Same idea, but they must use **the min.** possible integer of C
- YZ-**game Grundy number** $\Gamma_g^{YZ}(G)$: min number of colors s.t. Alice has a winning strat in the greedy YZ-coloring game
- We omit Z when it is "no one": $\Gamma_g^A(G) = \Gamma_g(G)$ is the original game Grundy number.

Literature - χ_g - Complexity

- [Bodlaender'91]: "*The complexity of Color Construction Game is an interesting open problem*"
- [Dunn et. al'15]: "*more than two decades later,* ***this question*** *remains open*".
- [Andres,Lock'19]: Introduced the variants χ_g^{YZ}. "*The question of PSPACE-hardness remains open for all the game variants mentioned above*", including the original one.
- [Costa,Soares,**Sampaio**'19]: Proved that $\chi_g(G)$ is PSPACE-hard, solving Bodlaender's 30-years question.

Our results-2

Complexity results

- Game chromatic n's χ_g^{YZ} are PSPACE-hard for all variants
- Game Grundy numbers Γ_g^{YZ} are PSPACE-hard for all variants
- All the related decision problems are PSPACE-Complete, even if the number of colors is the chromatic number

$\chi_g^B(G)$ is PSPACE-hard - Bob starts

Zhu'99 open question: Graph coloring game "*exhibits some strange properties*". Does Alice have a winning strategy with $k+1$ colors if she has one with k colors?

We define three decision problems for the graph coloring game:

- (Problem g_B-1) Given G and k: $\chi_g^B(G) \leq k$?
- (Problem g_B-2) Given G and k: Does Alice have a winning strategy with k colors?
- (Problem g_B-3) Given G and $\chi(G)$: $\chi_g^B(G) = \chi(G)$?

Problems g_B-1 and g_B-2 are generalizations of Problem g_B-3, when we know $\chi(G)$ - just take $k = \chi(G)$

Problem g_B-3 is PSPACE-hard $\rightarrow$ g_B-1 and g_B-2 are PSPACE-hard

χ_g^B: Reduction from POS-CNF$_B$

CNF formula, only positive literals, Bob and Alice alternate turns setting variables true or false. Alice wins if the formula is true. Bob wins if he selects (false) all variables of a clause.

Example

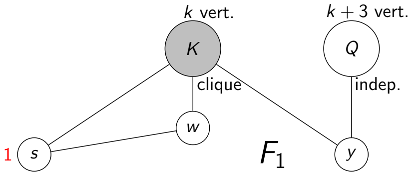
$(X_1 \vee X_2 \vee X_5) \wedge (X_1 \vee X_3 \vee X_5) \wedge (X_2 \vee X_4 \vee X_5) \wedge (X_3 \vee X_4 \vee X_5)$.
Bob has a winning strategy setting X_5 false first:

- X_1 True $\rightarrow X_4$ False;
- X_2 True $\rightarrow X_3$ False;
- X_4 True $\rightarrow X_1$ False;
- X_3 True $\rightarrow X_2$ False.

Good points

- POS-CNF$_B$ is PSPACE-Complete (POS-DNF [Shaefer'78])
- **Lemma:** If a player has a winning strategy in POS-CNF$_B$, the player also has a winning strat if the opponent pass turns.

χ_g^B: Important ingredient of the Reduction



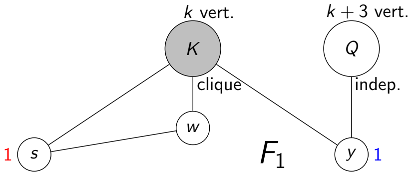
Lemma:

Suppose that Bob colored vertex s in his first move. Alice has a winning strategy in F_1 with $k+2$ colors **iff** she colors vertex y first with the same color of s.

Proof:

$k+2$ colors **iff** (y and s) or (y and w) have the same color.
If Alice colors y with the same color of s, she wins.
If Alice colors y with other color, Bob wins by coloring w with a different color. Otherwise, Bob colors the vertices of Q with distinct colors (starting with the color of s). If one of this colors in Q is not in K, he colors w with this color, and he wins.

χ_g^B: Important ingredient of the Reduction



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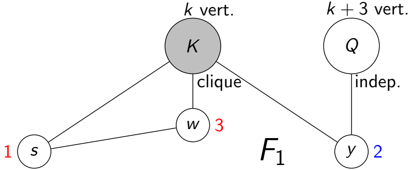

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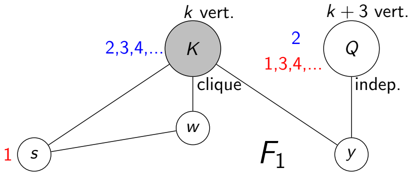

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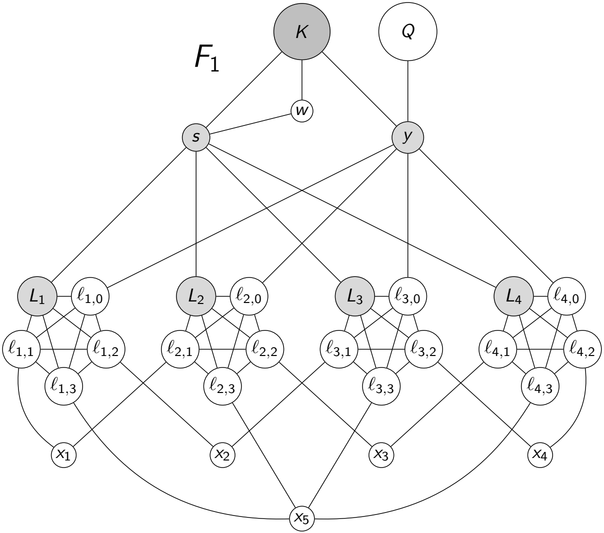
If Alice colors y with other color, Bob wins by coloring w with a different color. Otherwise, Bob colors the vertices of Q with distinct colors (starting with the color of s). If one of this colors in Q is not in K, he colors w with this color, and he wins.

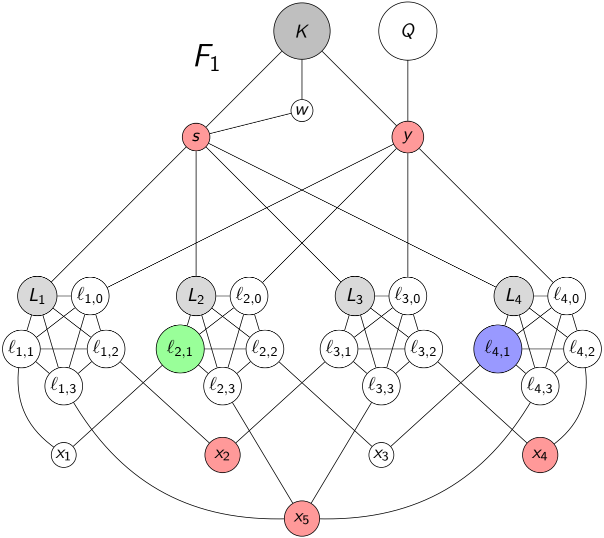
χ_g^B: Reduction from POS-CNF$_B$

$$(X_1 \vee X_2 \vee X_5) \wedge (X_1 \vee X_3 \vee X_5) \wedge (X_2 \vee X_4 \vee X_5) \wedge (X_3 \vee X_4 \vee X_5).$$



χ_g^B: Reduction from POS-CNF$_B$

$$(X_1 \vee X_2 \vee X_5) \wedge (X_1 \vee X_3 \vee X_5) \wedge (X_2 \vee X_4 \vee X_5) \wedge (X_3 \vee X_4 \vee X_5).$$



χ_g^{YZ} is PSPACE-hard for any Y, Z

Let $\chi_g^{A,A}(G)$, $\chi_g^{A,B}(G)$, $\chi_g^{B,A}(G)$ and $\chi_g^{B,B}(G)$: the minimum number of colors in C such that Alice has a winning strategy in $g_{A,A}$, $g_{A,B}$, $g_{B,A}$ and $g_{B,B}$, resp. For $Y, Z \in \{A, B\}$, let

- (Problem $g_{Y,Z}$-1) $\chi_g^{Y,Z}(G) \leq k$?
- (Problem $g_{Y,Z}$-2) Does Alice have a winning strategy in $g_{Y,Z}$ with k colors?
- (Problem $g_{Y,Z}$-3) $\chi_g^{Y,Z}(G) = \chi(G)$?

Corollary

For every $Y, Z \in \{A, B\}$, the decision problems $g_{Y,Z}$-1, $g_{Y,Z}$-2 and $g_{Y,Z}$-3 are PSPACE-complete.

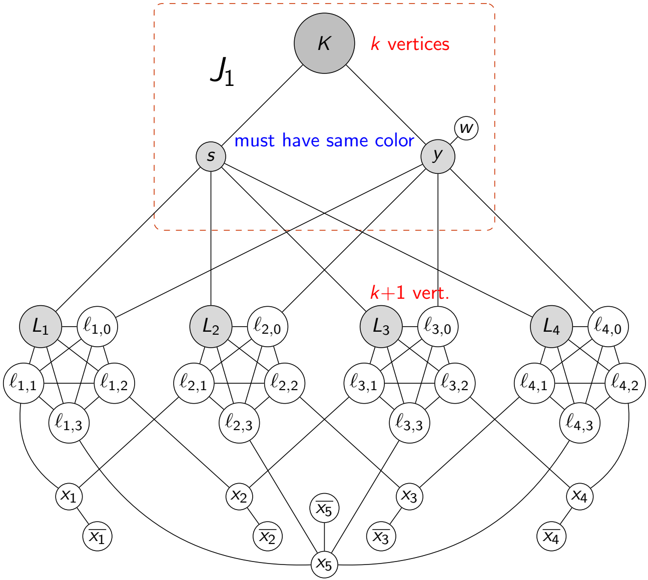
Proof.

Reduce from POS-CNF or POS-CNF$_B$ if $Y = A$ or $Y = B$, resp.
Since a winning strategy in both problems implies a winning strat if the opponent pass turns, we are almost done...

□

$\Gamma_g^B(G)$ is PSPACE-hard - Reduction POS-CNF$_B$

$(X_1 \vee X_2 \vee X_5) \wedge (X_1 \vee X_3 \vee X_5) \wedge (X_2 \vee X_4 \vee X_5) \wedge (X_3 \vee X_4 \vee X_5)$



Conclusion-2

Our results

- Game chromatic n's χ_g^{YZ} are PSPACE-hard for all variants
- Game Grundy numbers Γ_g^{YZ} are PSPACE-hard for all variants
- All the related decision problems are PSPACE-Complete, even if the number of colors is the chromatic number

THANK YOU !!

Connected graph coloring game

Connected game chromatic number ($\chi_{cg}^A(G) \geq \chi(G)$)

- Similar to the original graph coloring game: Alice starts and no one may pass turns
- But colored vertices must induce a connected subgraph
- **Connected game chromatic** number $\chi_{cg}^A(G)$: min number of colors s.t. Alice has a winning strategy in the connected graph coloring game

Literature - $\chi_{cg}^A(G)$

- Introduced by [Charpentier,Hocquard,Sopena,Zhu'19]
- [CHSZ'19]: Alice wins with 2 colors in bipartite graphs
- [CHSZ'19]: Alice wins with 5 colors in outerplanar graphs
- [Bradshaw'20]: There are outerplanar 2-trees with $\chi_{cg}^A(G) = 5$

Our results-3

Complexity results

- Connected game chromatic numbers χ_{cg}^{A} and χ_{cg}^{B} are PSPACE-hard
- Connected Grundy chromatic numbers Γ_{cg}^{A} and Γ_{cg}^{B} are PSPACE-hard
- All the related decision problems are PSPACE-Complete, even if the number of colors is the chromatic number

χ_{cg}^{A} is PSPACE-hard - Connected

Three decision problems for the Connected graph coloring game:

- (Problem cg_A-1) Given G and k: $\chi_{cg}^{A}(G) \leq k$?
- (Problem cg_A-2) Given G and k: Does Alice have a winning strategy with k colors?
- (Problem cg_A-3) Given G and $\chi(G)$: $\chi_{cg}^{A}(G) = \chi(G)$?

Problems cg_A-1 and cg_A-2 are generalizations of Problem cg_A-3, when we know $\chi(G)$ - just take $k = \chi(G)$

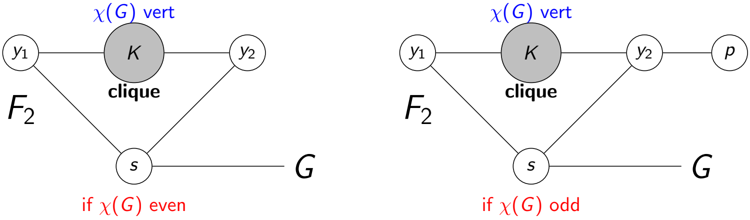
Problem cg_A-3 PSPACE-hard $\rightarrow$ cg_A-1 and cg_A-2 PSPACE-hard

Reduction from g_B-3: normal Coloring game when Bob starts.

- Given an instance $(G, \chi(G))$ of g_B-3,

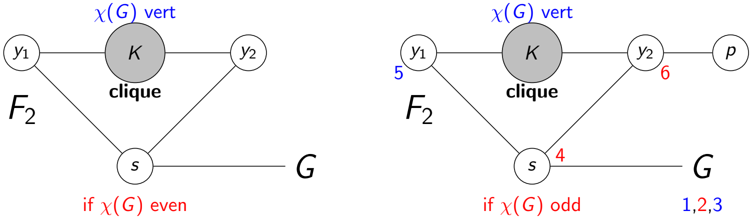
produce an instance $(G', \chi(G'))$ of cg_A-3.

χ_{cg}^A: PSPACE-hard reduction from g_B-3



- We may assume $|V(G)|$ odd
- $\chi_{cg}^A(G') = k + 1 \ \Rightarrow y_1$ and y_2 have the same color
- Alice must color y_1, y_2 or K first, since $|V(G) \cup \{s\}|$ even.
- If Alice has a winning strategy in g_B-3, she colors y_1 first and can guarantee that Bob is the first to play in G
- If Bob has a winning strategy in g_B-3, he can guarantee he is the first to play in G

χ_{cg}^{A}: PSPACE-hard reduction from g_B-3



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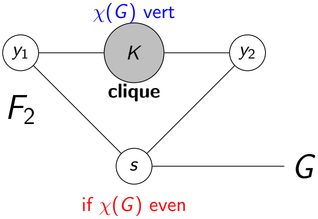

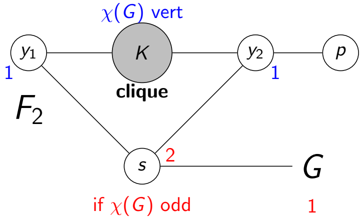

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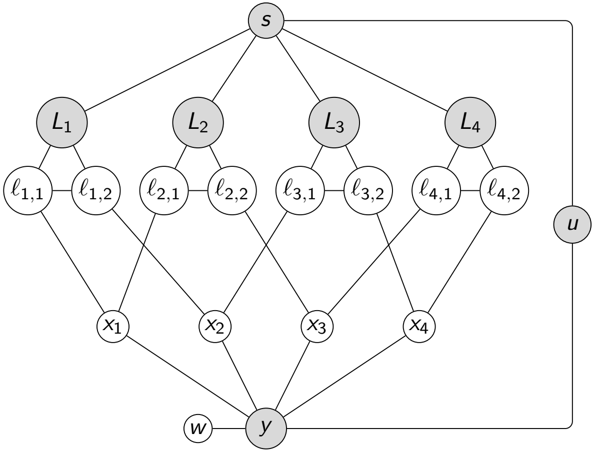
χ_{cg}^{B} is PSPACE-hard - reduction from POS-CNF

$$(X_1 \vee X_2) \quad \wedge \quad (X_1 \vee X_3) \quad \wedge \quad (X_2 \vee X_4) \quad \wedge \quad (X_3 \vee X_4)$$



Γ_{cg}^{A} is PSPACE-hard

Reduction from Γ_g^B.

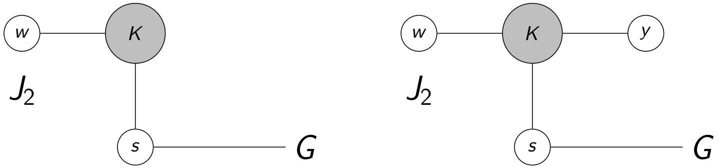
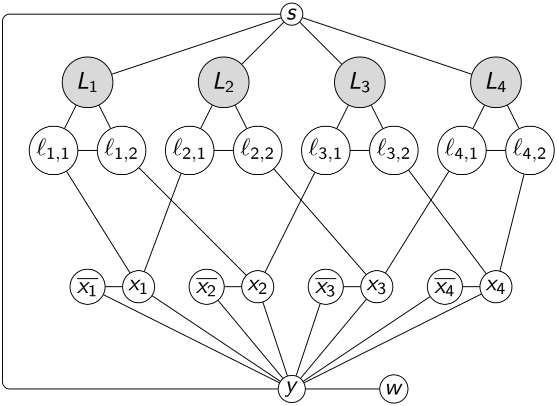


Figure: The reduction from the graph G. The left if $\chi(G)$ is odd or the right if $\chi(G)$ is even.

Γ_{cg}^{B} is PSPACE-hard

Reduction from POS-CNF$_B$.



Conclusion-3

Our results

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THANK YOU !!

Open Questions

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- Complexidade de todos esses problemas usando menos cores, por exemplo, 5 cores.
- Γ_{cg} (jogo conexo guloso) em classes de grafos.
- Variantes YZ normal ou gulosa em classes de grafos.
- Zhu question'99 em Variantes YZ não-gulosas: Alice tem estratégia vencedora com $k+1$ cores se possui com k cores?

THANK YOU !!